16.323 Lecture 15

Signals and System Norms

$\mathcal{H}_\infty$ Synthesis

Different type of optimal controller

• **Signal norms** we use norms to measure the size of a signal.

  - Three key properties of a norm:
    1. \( \|u\| \geq 0 \), and \( \|u\| = 0 \) iff \( u = 0 \)
    2. \( \|\alpha u\| = |\alpha|\|u\| \) \( \forall \) scalars \( \alpha \)
    3. \( \|u + v\| \leq \|u\| + \|v\| \)

• **Key signal norms**

  - 2-norm of \( u(t) \) – *Energy of the signal*
    \[
    \|u(t)\|_2 = \left( \int_{-\infty}^{\infty} u^2(t) dt \right)^{1/2}
    \]

  - \( \infty \)-norm of \( u(t) \) – *maximum value over time*
    \[
    \|u(t)\|_\infty = \max_t |u(t)|
    \]

  - Other useful measures include the *Average power*
    \[
    \text{pow}(u(t)) = \left( \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} u^2(t) dt \right)^{1/2}
    \]

    \( u(t) \) is called a *power signal* if \( \text{pow}(u(t)) < \infty \)
• **System norms** Consider the system with dynamics \( y = G(s)u \)
  
  – Assume \( G(s) \) stable, LTI transfer function matrix
  – \( g(t) \) is the associated impulse response matrix (causal).

• \( \mathcal{H}_2 \) norm for the system: (LQG problem)

\[
\|G\|_2 = \left( \frac{1}{2\pi} \int_{-\infty}^{\infty} \text{trace}[G^H(j\omega)G(j\omega)]d\omega \right)^{1/2} \\
= \left( \int_{0}^{\infty} \text{trace}[g^T(\tau)g(\tau)]d\tau \right)^{1/2}
\]

Two interpretations:

– For SISO: energy in the output \( y(t) \) for a unit impulse input \( u(t) \).
– For MIMO \(^{27}\): apply an impulsive input separately to each actuator and measure the response \( z_i \), then

\[
\|G\|_2^2 = \sum_i \|z_i\|_2^2
\]

– Can also interpret as the expected RMS value of the output in response to unit-intensity white noise input excitation.

• **Key point:** Can show that

\[
\|G\|_2 = \left( \frac{1}{2\pi} \int_{-\infty}^{\infty} \sum_i \sigma_i^2[G(j\omega)]d\omega \right)^{1/2}
\]

– Where \( \sigma_i[G(j\omega)] \) is the \( i \)th singular value\(^{28} \) \(^{29}\) of the system \( G(s) \) evaluated at \( s = j\omega \)

– \( \mathcal{H}_2 \) norm concerned with overall performance \( (\sum_i \sigma_i^2) \) over all frequencies

\(^{27}\)ZDG114
\(^{28}\)http://mathworld.wolfram.com/SingularValueDecomposition.html
\(^{29}\)http://en.wikipedia.org/wiki/Singular_value_decomposition

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• $\mathcal{H}_\infty$ norm for the system:

$$\|G(s)\|_\infty = \sup_\omega \sigma[G(j\omega)]$$

Interpretation:

- $\|G(s)\|_\infty$ is the “energy gain” from the input $u$ to output $y$

$$\|G(s)\|_\infty = \max_{u(t) \neq 0} \int_0^\infty y^T(t)y(t)dt$$

$$\|G(s)\|_\infty = \max_{u(t) \neq 0} \int_0^\infty u^T(t)u(t)dt$$

- Achieve this maximum gain using a worst case input signal that is essentially a sinusoid at frequency $\omega^*$ with input direction that yields $\sigma[G(j\omega^*)]$ as the amplification.

![Graphical test for the $\|G\|_\infty$.](image)

Figure 15.1: Graphical test for the $\|G\|_\infty$.

• Note that we now have

1. Signal norm $\|u(t)\|_\infty = \max |u(t)|$
2. Vector norm $\|x\|_\infty = \max_i |x_i|$
3. System norm $\|G(s)\|_\infty = \max_\omega \sigma[G(j\omega)]$

We use the same symbol $\| \cdot \|_\infty$ for all three, but there is typically no confusion, as the norm to be used is always clear by the context.
So $\mathcal{H}_\infty$ is concerned primarily with the peaks in the frequency response, and the $\mathcal{H}_2$ norm is concerned with the overall response.

The $\mathcal{H}_\infty$ norm satisfies the submultiplicative property

$$\|GH\|_\infty \leq \|G\|_\infty \cdot \|H\|_\infty$$

- Will see that this is an essential property for the robustness tests
- Does not hold in general for $\|GH\|_2$

Reference to $\mathcal{H}_\infty$ control is that we would like to design a stabilizing controller that ensures that the peaks in the transfer function matrix of interest are knocked down.

\[ \text{e.g. want } \max_\omega \sigma[T(j\omega)] \equiv \|T(s)\|_\infty < 0.75 \]

Reference to $\mathcal{H}_2$ control is that we would like to design a stabilizing controller that reduces the $\|T(s)\|_2$ as much as possible.

- Note that $\mathcal{H}_2$ control and LQG are the same thing.
• Assume that \( G(s) = C(sI - A)^{-1}B + D \) with \( \Re \lambda(A) < 0 \), i.e. \( G(s) \) stable.

• \( \mathcal{H}_2 \) norm: requires a strictly proper system \( D = 0 \)

\[
\begin{align*}
\dot{x} &= Ax + Bu \\
y &= Cx
\end{align*}
\]

– Define:

**Observability Gramian** \( P_o \)

\[
A^T P_o + P_o A + C^T C = 0 \iff P_o = \int_0^\infty e^{A^T t} C^T C e^{At} \, dt
\]

**Controllability Gramian** \( P_c \)

\[
A P_c + P_c A^T + B B^T = 0 \iff P_c = \int_0^\infty e^{At} B B^T e^{A^T t} \, dt
\]

then

\[
\|G\|^2_2 = \text{trace} \left( B^T P_o B \right) = \text{trace} \left( C P_c C^T \right)
\]

**Proof:** use the impulse response of the system \( G(s) \) and evaluate the time-domain version of the norm.

• \( \mathcal{H}_\infty \) norm: Define the **Hamiltonian matrix**

\[
H = \begin{bmatrix}
A + B(\gamma^2 I - D^T D)^{-1} D^T C & B(\gamma^2 I - D^T D)^{-1} B^T \\
- C^T (I + D(\gamma^2 I - D^T D)^{-1} D^T) C & -(A + B(\gamma^2 I - D^T D)^{-1} D^T C)^T
\end{bmatrix}
\]

– Then \( \|G(s)\|_\infty < \gamma \) iff \( \overline{\sigma(D)} < \gamma \) and \( H \) has no eigenvalues on the \( j\omega \)-axis.

– Graphical test \( \max_\omega \overline{\sigma[G(j\omega)]} < \gamma \) replaced with eigenvalue test.
Note that it is **not easy** to find $\|G\|_\infty$ directly using the state space techniques

- It is easy to check if $\|G\|_\infty < \gamma$
- So we just keep changing $\gamma$ to find the smallest value for which we can show that $\|G\|_\infty < \gamma$ (called $\gamma_{\text{min}}$)

$\Rightarrow$ Bisection search algorithm.

**Bisection search algorithm**

1. Select $\gamma_u$, $\gamma_l$ so that $\gamma_l \leq \|G\|_\infty \leq \gamma_u$

2. Test $(\gamma_u - \gamma_l)/\gamma_l < \text{TOL}$.
   - **Yes** $\Rightarrow$ Stop ($\|G\|_\infty \approx \frac{1}{2}(\gamma_u + \gamma_l)$)
   - **No** $\Rightarrow$ go to step 3.

3. With $\gamma = \frac{1}{2}(\gamma_l + \gamma_u)$, test if $\|G\|_\infty < \gamma$ using $\lambda_i(H)$

4. If $\lambda_i(H) \in j\mathcal{R}$, then set $\gamma_l = \gamma$ (test value too low), otherwise set $\gamma_u = \gamma$ and go to step 2.
Note that we can use the state space tests to analyze the weighted tests that we developed for robust stability

- For example, we have seen the value in ensuring that the sensitivity remains smaller than a particular value
  \[ \sigma [ W_i S(j\omega) ] < 1 \quad \forall \ \omega \]

- We can test this by determining if \( \| W_i(s) S(s) \|_\infty < 1 \)
  - Use state space models of \( G_c(s) \) and \( G(s) \) to develop a state space model of
    \[
    S(s) := \begin{bmatrix} A_s & B_s \\ C_s & 0 \end{bmatrix}
    \]
  - Augment these dynamics with the (stable, min phase) \( W_i(s) \) to get a model of \( W_i(s) S(s) \)
    \[
    W_i(s) := \begin{bmatrix} A_w & B_w \\ C_w & 0 \end{bmatrix}
    \]
    \[
    W_i(s) S(s) := \begin{bmatrix} A_s & 0 & B_s \\ B_w C_s & A_w & 0 \\ 0 & C_w & 0 \end{bmatrix}
    \]
  - Now compute the \( \mathcal{H}_\infty \) norm of the combined system \( W_i(s) S(s) \).
Riccati Equation Tests

- Note that, with $D = 0$, the $\mathcal{H}_\infty$ Hamiltonian matrix becomes
  \[
  H = \begin{bmatrix}
  A & \frac{1}{\gamma^2}BB^T \\
  -C^T & -A^T
  \end{bmatrix}
  \]

  - Know that $\|G\|_\infty < \gamma$ iff $H$ has no eigenvalues on the $j\omega$-axis.
  - Equivalent test is if there exists a $X \geq 0$ such that
    \[
    A^T X + XA + C^T C + \frac{1}{\gamma^2} X BB^T X = 0
    \]
    and $A + \frac{1}{\gamma^2} BB^T X$ is stable.
  - So there is a direction relationship between the Hamiltonian matrix $H$ and the algebraic Riccati Equation (ARE)

- Aside: Compare this ARE with the one that we would get if we used this system in an LQR problem:
  \[
  A^T P + PA + C^T C - \frac{1}{\rho} P BB^T P = 0
  \]
  - If $(A, B, C)$ stabilizable/detectable, then will always get a solution for the LQR ARE.
  - Sign difference in quadratic term of the $\mathcal{H}_\infty$ ARE makes this equation harder to satisfy. Consistent with the fact that we could have $\|G\|_\infty > \gamma \Rightarrow$ no solution to the $\mathcal{H}_\infty$ ARE.
  - The two Riccati equations look similar, but with the sign change, the solutions can behave very differently.

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For the synthesis problem, we typically define a generalized version of the system dynamics

\[
P(s) = \begin{bmatrix} P_{zw}(s) & P_{zu}(s) \\ P_{yw}(s) & P_{yu}(s) \end{bmatrix}
\]

contains the plant \( G(s) \) and all performance and uncertainty weights

Signals:
- \( z \) Performance output
- \( w \) Disturbance/ref inputs
- \( y \) Sensor outputs
- \( u \) Actuator inputs

Generalized plant:

\[
(z \atop w)_{CL} = P_{zw} + P_{zu}G_c(I - P_{yu}G_c)^{-1}P_{yw} \\
\equiv F_l(P, G_c)
\]

called a (lower) **Linear Fractional Transformation** (LFT).
• **Design Objective:** Find $G_c(s)$ to stabilize the closed-loop system and minimize $\|F_i(P, G_c)\|_\infty$.

• Hard problem to solve, so we typically consider a suboptimal problem:
  - Find $G_c(s)$ to satisfy $\|F_i(P, G_c)\|_\infty < \gamma$
  - Then use bisection (called a $\gamma$ iteration) to find the smallest value ($\gamma_{opt}$) for which $\|F_i(P, G_c)\|_\infty < \gamma_{opt}$

  $\Rightarrow$ hopefully get that $G_c$ approaches $G_{c_{opt}}$

• Consider the suboptimal $\mathcal{H}_\infty$ synthesis problem:  

$$
\begin{align*}
\text{Find } & G_c(s) \text{ to satisfy } \|F_i(P, G_c)\|_\infty < \gamma \\
\text{where we assume that:} & \\
1. & (A, B_u, C_y) \text{ is stabilizable/detectable (essential)} \\
2. & (A, B_w, C_z) \text{ is stabilizable/detectable (essential)} \\
3. & D_{zu}^T \begin{bmatrix} C_z & D_{zu} \end{bmatrix} = \begin{bmatrix} 0 & I \end{bmatrix} \text{ (simplify/essential)} \\
4. & \begin{bmatrix} B_w \\ D_{yw} \end{bmatrix} D_{yw}^T = \begin{bmatrix} 0 \\ I \end{bmatrix} \text{ (simplify/essential)}
\end{align*}
$$

• Note that we will not cover all the details of the solution to this problem – it is well covered in the texts.
There exists a stabilizing $G_c(s)$ such that $\|F_i(P,G_c)\|_\infty < \gamma$ iff

1. $\exists X \geq 0$ that solves the ARE
   
   $$A^TX + XA + C_z^TC_z + X(\gamma^{-2}B_wB_w^T - B_uB_u^T)X = 0$$
   
   and $\Re \lambda_i [A + (\gamma^{-2}B_wB_w^T - B_uB_u^T)X] < 0 \ \forall \ i$

2. $\exists Y \geq 0$ that solves the ARE
   
   $$AY + YA^T + B_w^TB_w + Y(\gamma^{-2}C_z^TC_z - C_y^TC_y)Y = 0$$
   
   and $\Re \lambda_i [A + Y(\gamma^{-2}C_z^TC_z - C_y^TC_y)] < 0 \ \forall \ i$

3. $\rho(XY) < \gamma^2$

$\rho$ is the spectral radius ($\rho(A) = \max_i |\lambda_i(A)|$).

Given these solutions, the central $\mathcal{H}_\infty$ controller is given by

$$G_c(s) := \begin{bmatrix}
A + (\gamma^{-2}B_wB_w^T - B_uB_u^T)X - ZY C_y^TC_y & ZY C_y^T \\
-B_u^TX & 0
\end{bmatrix}$$

where $Z = (I - \gamma^{-2}XY)^{-1}$

- Central controller has as many states as the generalized plant.

Note that this design does not decouple as well as the regulator/estimator for LQG
• Basic assumptions:
(A1) \((A, B_u, C_y)\) is stabilizable/detectable
(A2) \((A, B_w, C_z)\) is stabilizable/detectable
(A3) \(D_{zu}^T \begin{bmatrix} C_z & D_{zu} \end{bmatrix} = \begin{bmatrix} 0 & I \end{bmatrix}\) (scaling and no cross-coupling)
(A4) \( \begin{bmatrix} B_w \\ D_{yw} \end{bmatrix} \begin{bmatrix} D_{yu}^T \end{bmatrix} = \begin{bmatrix} 0 & I \end{bmatrix}\) (scaling and no cross-coupling)

• The restrictions that \(D_{zw} = 0\) and \(D_{yu} = 0\) are weak, and can easily be removed (the codes handle the more general \(D\) case).

• (A1) is required to ensure that it is even possible to get a stabilizing controller.

• Need \(D_{zu}\) and \(D_{yw}\) to have full rank to ensure that we penalize control effort (A3) and include sensor noise (A4)
  \(\Rightarrow\) Avoids singular case with infinite bandwidth controllers.
  \(\Rightarrow\) Often where you will have the most difficulties initially.

• Typically will see two of the assumptions written as:
  \((Ai)\) \(\begin{bmatrix} A - j\omega I & B_u \\ C_z & D_{zu} \end{bmatrix}\) has full column rank \(\forall \omega\)
  \((Aii)\) \(\begin{bmatrix} A - j\omega I & B_w \\ C_y & D_{yw} \end{bmatrix}\) has full row rank \(\forall \omega\)
  – These ensure that there are no \(j\omega\)-axis zeros in the \(P_{zu}\) or \(P_{yw}\) TF’s
  – cannot have the controller canceling these, because that design would not internally stabilize the closed-loop system.
  – But with assumptions (A3) and (A4) given above, can show that \(A(i)\) and \(A(ii)\) are equivalent to our assumption (A2).
Simple Design Example

\[
G = \frac{200}{(0.05s + 1)^2(10s + 1)}
\]

- Note that we have 1 input \(r\) and two performance outputs - one that penalizes the sensitivity \(S(s)\) of the system, and the other that penalizes the control effort used.

- Easy to show (see next page) that the closed-loop is:

\[
\begin{bmatrix}
  z_1 \\
  z_2
\end{bmatrix} = \begin{bmatrix}
  W_s S \\
  W_u G_c S
\end{bmatrix} r
\]

where, in this case, the input \(r\) acts as the "disturbance input" \(w\) to the generalized system.

- To achieve good low frequency tracking and a crossover frequency of about 10 rad/sec, pick

\[
W_s = \frac{s/1.5 + 10}{s + (10) \cdot (0.0001)} \quad W_u = 1
\]
Generalized system in this case:

\[
\begin{align*}
z_1 &= W_s(s)(r - Gu) \\
z_2 &= W_u u \\
e &= r - Gu \\
u &= G_c e
\end{align*}
\]

\[
P(s) = \begin{bmatrix}
W_s(s) & -W_s(s)G(s) \\
0 & W_u(s) \\
1 & -G(s)
\end{bmatrix}
\]

\[
P_{CL} = F_l(P, G_c)
= \begin{bmatrix}
W_s \\
0
\end{bmatrix} + \begin{bmatrix}
-W_sG \\
W_u
\end{bmatrix} G_c (I + G G_c)^{-1} 1
= \begin{bmatrix}
W_s - W_s G G_c S \\
W_u G_c S
\end{bmatrix} = \begin{bmatrix}
W_s S \\
W_u G_c S
\end{bmatrix}
\]
• In state space form, let

\[ G(s) := \begin{bmatrix} A & B \\ C & 0 \end{bmatrix} \quad W_s(s) := \begin{bmatrix} A_w & B_w \\ C_w & D_w \end{bmatrix} \] \quad W_u = 1

\[ \begin{align*}
\dot{x} &= Ax + Bu \\
\dot{x}_w &= A_w x_w + B_w e = A_w x_w + B_w r - B_w C x \\
z_1 &= C_w x_w + D_w e = C_w x_w + D_w r - D_w C x \\
z_2 &= W_u u \\
e &= r - C x
\end{align*} \]

\[ P(s) := \begin{bmatrix}
A & 0 & 0 & B \\
-B_w C & A_w & B_w & 0 \\
-D_w C & C_w & D_w & 0 \\
0 & 0 & 0 & W_u \\
-C & 0 & 1 & 0
\end{bmatrix} \]

• Now use the mu-tools code to solve for the controller. (Could also have used the robust control toolbox code).

```matlab
A=[A,
zeros(n1,n2);-Bsw*G Cg Asw];
Bw=[zeros(n1,1);Bsw];
Bu=[Bg;zeros(n2,1)];
Cz=[-Dsw*G Csw;zeros(1,n1+n2)];
Cy=[-G C zeros(1,n2)];
Dzw=[Dsw;0];
Dzu=[0;1];
Dyw=[1];
Dyu=0;
P=pck(A,[Bw Bu],[Cz;Cy],[Dzw Dzu;Dyw Dyu]);
% call hinf to find Gc (mu toolbox)
[Gc,G,gamma]=hinfsyn(P,1,1,0.1,20,.001);
```

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• Results from the $\gamma$-iteration showing whether we pass or fail the various $X, Y, \rho(XY)$ tests as we keep searching over $\gamma$, starting at the initial bound of 20.

Resetting value of Gamma min based on $D_{11}, D_{12}, D_{21}$ terms

Test bounds: $0.6667 < \gamma \leq 20.0000$

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<th>yinf_eig</th>
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<td>p</td>
</tr>
</tbody>
</table>

> 1.271 9.1e+000 -1.2e+004# 1.0e-003 -4.5e-010 0.0000  f
| 1.573  | 9.3e+000 | 7.3e-008 | 1.0e-003 | 0.0e+000 | 0.0000  | p   |
| 1.422  | 9.2e+000 | 7.6e-008 | 1.0e-003 | 0.0e+000 | 0.0000  | p   |

> 1.346 9.2e+000 -6.4e+004# 1.0e-003 0.0e+000 0.0000  f
| 1.384  | 9.2e+000 | 7.7e-008 | 1.0e-003 | 0.0e+000 | 0.0000  | p   |

> 1.365 9.2e+000 -1.9e+006# 1.0e-003 0.0e+000 0.0000  f
| 1.375  | 9.2e+000 | 7.7e-008 | 1.0e-003 | -4.5e-010 0.0000  p
| 1.370  | 9.2e+000 | 7.7e-008 | 1.0e-003 | 0.0e+000 0.0000  p
| 1.368  | 9.2e+000 | 7.7e-008 | 1.0e-003 | 0.0e+000 0.0000  p
| 1.366  | 9.2e+000 | 7.7e-008 | 1.0e-003 | 0.0e+000 0.0000  p

> 1.366 9.2e+000 -1.3e+007# 1.0e-003 0.0e+000 0.0000  f

Gamma value achieved: 1.3664

• Since $\gamma_{\text{min}} = 1.3664$, this indicates that we did not meet the desired goal of $|S| < 1/|W_s|$ (can only say that $|S| < 1.3664/|W_s|$).
  – Confirmed by the plot, which shows that we just fail the test (blue line passes above magenta)

• But note that, even though this design fails the sensitivity weight - we still get pretty good performance
  – For performance problems, can think of the objective of getting $\gamma_{\text{min}} < 1$ as a “design goal” → it is “not crucial”
  – Use $W_{iu}$ to tune the control design

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Figure 15.3: Visualization of the weighted sensitivity tests.

Figure 15.4: Time response of controller that yields $\gamma_{\min} = 1.3664$. 
General LQG Problem

- Can also put LQG ($H_2$) design into this generalized framework.\(^{31}\)

- Define the dynamics

\[
\begin{align*}
\dot{x} &= Ax + Bu + w_d \\
y &= Cx + w_n
\end{align*}
\]

where

\[
E \left\{ \begin{bmatrix} w_d(t) \\ w_n(t) \end{bmatrix} \begin{bmatrix} w_d^T(\tau) \\ w_n^T(\tau) \end{bmatrix} \right\} = \begin{bmatrix} W & 0 \\ 0 & V \end{bmatrix} \delta(t - \tau)
\]

- LQG problem is to find controller $u = G_c(s)y$ that minimizes

\[
J = E \left\{ \lim_{T \to \infty} \frac{1}{T} \int_0^T (x^T R_{xx} x + u^T R_{uu} u) dt \right\}
\]

- To put this problem in the general framework, define

\[
z = \begin{bmatrix} R_{xx}^{1/2} & 0 \\ 0 & R_{uu}^{1/2} \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} w_d \\ w_n \end{bmatrix} = \begin{bmatrix} W^{1/2} & 0 \\ 0 & V^{1/2} \end{bmatrix} w
\]

where $w$ is a unit intensity white noise process.

- With $z = F_i(P, G_c)w$, the LQG cost function can be rewritten as

\[
J = E \left\{ \lim_{T \to \infty} \frac{1}{T} \int_0^T z^T(t) z(t) dt \right\} = \| F_i(P, G_c) \|^2_2
\]

- In this case the generalized plant matrix is

\[
P(s) := \begin{bmatrix}
A & W^{1/2} & 0 & B \\
\frac{1}{2} R_{xx}^{1/2} & 0 & 0 & 0 \\
0 & 0 & 0 & R_{uu}^{1/2} \\
C & 0 & V^{1/2} & 0
\end{bmatrix}
\]

\(^{31}\text{SP365}\)

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Given these solutions, the central $\mathcal{H}_\infty$ controller is given by

$$G_c(s) := \begin{bmatrix} A + (\gamma^{-2}B_wB_w^T - B_uB_u^T)X - ZYC_y^TC_y & ZYC_y^T \\ -B_u^TX & 0 \end{bmatrix}$$

where $Z = (I - \gamma^{-2}YX)^{-1}$

Can develop a further interpretation of this controller if we rewrite the dynamics as:

$$\dot{x} = Ax + \gamma^{-2}B_wB_w^T\hat{x} - B_uB_u^TX\hat{x} - ZYC_y^TC_y\hat{x} + ZYC_y^Ty$$

$$u = -B_u^TX\hat{x}$$

$$\Rightarrow \dot{x} = Ax + B_w[\gamma^{-2}B_w^T\hat{x}] + B_u[-B_u^TX\hat{x}] + ZYC_y^T[y - C_y\hat{x}]$$

$$\Rightarrow \dot{x} = Ax + B_w[\gamma^{-2}B_w^T\hat{x}] + Bu + L[y - C_y\hat{x}]$$

looks very similar to Kalman Filter developed for LQG controller.

The difference is that we have an additional input $\dot{w}_{\text{worst}} = \gamma^{-2}B_w^T\hat{x}$ that enters through $B_w$.

$w_{\text{worst}}$ is an estimate of worst-case disturbance to the system.

Finally, note that a separation rule does exist for the $\mathcal{H}_\infty$ controller. But we will not discuss it.
Code: $\mathcal{H}_\infty$ Synthesis

1  \% Hinf example
2  \% 16.323 MIT Spring 2007
3  \% Jon How
4  \%
5  set(0,'DefaultAxesFontName','arial')
6  set(0,'DefaultAxesFontSize',16)
7  set(0,'DefaultTextFontName','arial')
8  set(0,'DefaultTextFontSize',20)
9  clear all
10 if ~exist('yprev')
11     yprev=[1 1]';
12     tprev=[0 1]';
13     Sensprev=[1 1]';
14     fprev=[.1 100]';
15 end
16 \%
17 \% define plant
18 \[\begin{align*}
19 & [\mathbf{A}_g,\mathbf{B}_g,\mathbf{C}_g,\mathbf{D}_g] = \text{tf2ss}(200, \text{conv}(\text{conv}([0.05 1],[0.05 1]),[10 1])); \\
20 & \mathbf{G}_0 = \text{ss}(\mathbf{A}_g,\mathbf{B}_g,\mathbf{C}_g,\mathbf{D}_g);
21 \end{align*} \]
22 \%
23 \% define sensitivity weight
24 M=1.5; \%B=10; \%A=1e-4;
25 [\mathbf{A}_s,\mathbf{B}_s,\mathbf{C}_s,\mathbf{D}_s] = \text{tf2ss}([1/M \, \mathbf{w}_B], [1 \, \mathbf{w}_B \cdot \mathbf{A}]);
26 \mathbf{W}_s = \text{ss}(\mathbf{A}_s,\mathbf{B}_s,\mathbf{C}_s,\mathbf{D}_s);
27 \%
28 \% form augmented P dynamics
29 \%
30 n1=size(\mathbf{A}_g,1);
31 n2=size(\mathbf{A}_s,1);
32 \mathbf{A} = [\mathbf{A}_g \, \text{zeros}(n1,n2); \mathbf{B}_s];
33 \mathbf{B}_w = [\text{zeros}(n1,1); \mathbf{B}_s];
34 \mathbf{B}_u = [\mathbf{B}_g; \text{zeros}(n2,1)];
35 \mathbf{C}_z = [\mathbf{D}_s \mathbf{C}_s; \text{zeros}(1,n1+n2)];
36 \mathbf{C}_y = [\mathbf{C}_g \, \text{zeros}(1,n2)];
37 \mathbf{D}_w = [\mathbf{D}_s; 0];
38 \mathbf{D}_u = [0; \mathbf{W}_s];
39 \mathbf{D}_y = [1];
40 \mathbf{D}_u = 0;
41 \%
42 \%
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67 \%

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loglog(f, abs(CLWS), 'r-', 'LineWeight', 2)
loglog(fprev, abs(Sensprev), 'r-.', 'LineWidth', 2)
legend('S', '1/W_s', 'W_sS', 'Location', 'SouthEast')
hold off
xlabel('Freq (rad/sec)')
ylabel('Magnitude')
grid

print -depsc hinf1.eps; jpdf('hinf1')

na = size(Ag, 1);
nac = size(ac, 1);
Acl = [Ag Bg*cc; -bc*Cg ac]; Bcl = [zeros(na, 1); bc]; Ccl = [Cg zeros(1, nac)]; Dcl = 0;
Gcl = ss(Acl, Bcl, Ccl, Dcl);
[y, t] = step(Gcl, 1);

figure(2); clf
plot(t, y, 'LineWidth', 2)
hold on; plot(tprev, yprev, 'r--', 'LineWidth', 2); hold off
xlabel('Time sec')
ylabel('Step response')

print -depsc hinf12.eps; jpdf('hinf12')

yprev = y;
tprev = t;
Sensprev = Sens;
fprev = f;