16.333: Lecture #3

Frame Rotations

Euler Angles

Quaternions
Euler Angles

- For general applications in 3D, often need to perform 3 separate rotations to relate our “inertial frame” to our “body frame”
  - Especially true for aircraft problems

- There are many ways to do this set of rotations - with the variations be based on the order of the rotations
  - All would be acceptable
  - Some are more commonly used than others

- Standard: start with the body frame \((x, y, z)\) aligned with the inertial \((X, Y, Z)\), and then perform 3 rotations to re-orient the body frame.

1. Rotate by \(\psi\) about \(Z \Rightarrow x', y', z'\)
2. Rotate by \(\theta\) about \(y' \Rightarrow x'', y'', z''\)
3. Rotate by \(\phi\) about \(x'' \Rightarrow x, y, z\)

Euler angles:
- \(\psi \sim\) Heading/yaw
- \(\theta \sim\) Pitch
- \(\phi \sim\) Roll
• Can write these rotations in a convenient form:

\[
\begin{bmatrix}
  x' \\
y' \\
z'
\end{bmatrix} = \begin{bmatrix}
  c\psi & s\psi & 0 \\
  -s\psi & c\psi & 0 \\
  0 & 0 & 1
\end{bmatrix} \begin{bmatrix}
  X \\
  Y \\
  Z
\end{bmatrix} = T_3(\psi) \begin{bmatrix}
  X \\
  Y \\
  Z
\end{bmatrix}
\]

\[
\begin{bmatrix}
  x'' \\
y'' \\
z''
\end{bmatrix} = \begin{bmatrix}
  c\theta & 0 & -s\theta \\
  0 & 1 & 0 \\
  s\theta & 0 & c\theta
\end{bmatrix} \begin{bmatrix}
  x' \\
y' \\
z'
\end{bmatrix} = T_2(\theta) \begin{bmatrix}
  x' \\
y' \\
z'
\end{bmatrix}
\]

\[
\begin{bmatrix}
  x \\
y \\
z
\end{bmatrix} = \begin{bmatrix}
  1 & 0 & 0 \\
  0 & c\phi & s\phi \\
  0 & -s\phi & c\phi
\end{bmatrix} \begin{bmatrix}
  x'' \\
y'' \\
z''
\end{bmatrix} = T_1(\phi) \begin{bmatrix}
  x'' \\
y'' \\
z''
\end{bmatrix}
\]

which combines to give:

\[
\begin{bmatrix}
  x \\
y \\
z
\end{bmatrix} = T_1(\phi)T_2(\theta)T_3(\psi) \begin{bmatrix}
  X \\
  Y \\
  Z
\end{bmatrix}
\]

\[
= \begin{bmatrix}
  c\theta c\psi & c\theta s\psi & -s\theta \\
  -c\phi s\psi + s\phi s\theta c\psi & c\phi c\psi + s\phi s\theta s\psi & s\phi c\theta \\
  s\phi s\psi + c\phi s\theta c\psi & -s\phi c\psi + c\phi s\theta s\psi & c\phi c\theta
\end{bmatrix} \begin{bmatrix}
  X \\
  Y \\
  Z
\end{bmatrix}
\]

• Note that the order that these rotations are applied matters and will greatly change the answer – matrix multiplies of $T_i$ must be done consistently.
• To get the angular velocity in this case, we have to include three terms:
  ① $\dot{\psi}$ about $Z$
  ② $\dot{\theta}$ about $y'$
  ③ $\dot{\phi}$ about $x''$

which we combine to get $\vec{\omega}$

• Want to write $\vec{\omega}$ in terms of its components in final frame (body)
  - Use the rotation matrices

• Example: rotate $\dot{\psi}$ about $Z \equiv z'$
  - In terms of $X, Y, Z$, frame rotation rate has components
    \[
    \begin{bmatrix}
      0 \\
      0 \\
      \dot{\psi}
    \end{bmatrix},
    \]
    which is the same as in frame $x', y', z'$
  - To transform a vector from $x', y', z'$ to $x, y, z$, need to use $T_1(\phi)T_2(\theta)$
  - Similar operation for $\dot{\theta}$ about $y' \equiv y'' \Rightarrow$ use $T_1(\phi)$ on
    \[
    \begin{bmatrix}
      0 \\
      \dot{\phi} \\
      0
    \end{bmatrix}
    \]

• Final result:
  \[
  \begin{bmatrix}
    \omega_x \\
    \omega_y \\
    \omega_z
  \end{bmatrix} = T_1(\phi)T_2(\theta)
  \begin{bmatrix}
    0 \\
    \dot{\psi} \\
    0
  \end{bmatrix}
  + T_1(\phi)
  \begin{bmatrix}
    0 \\
    \dot{\theta} \\
    0
  \end{bmatrix}
  + \begin{bmatrix}
    \dot{\phi} \\
    0 \\
    0
  \end{bmatrix}
  \]
- Visualization: Can write

\[ \vec{\omega} = \vec{\phi} + \vec{\theta} + \vec{\psi} \]

But \( \vec{\phi}, \vec{\theta}, \vec{\psi} \) do not form a mutually orthogonal triad.

Need to form the orthogonal projections onto the body frame \( x \),

\[
\omega_b = \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} = T_1(\phi)T_2(\theta) \begin{bmatrix} 0 \\ 0 \\ \dot{\psi} \end{bmatrix} + T_1(\phi) \begin{bmatrix} 0 \\ \dot{\theta} \\ 0 \end{bmatrix} + \begin{bmatrix} \dot{\phi} \\ 0 \\ 0 \end{bmatrix}
\]

- Final form

\[
\begin{align*}
\omega_x &= \dot{\phi} - \dot{\psi} \sin \theta \\
\omega_y &= \dot{\theta} \cos \phi + \dot{\psi} \cos \theta \sin \phi \\
\omega_z &= -\dot{\theta} \sin \phi + \dot{\psi} \cos \theta \cos \phi
\end{align*}
\]

- With inverse:

\[
\begin{align*}
\dot{\phi} &= \omega_x + [\omega_y \sin \phi + \omega_z \cos \phi] \tan \theta \\
\dot{\theta} &= \omega_y \cos \phi - \omega_z \sin \phi \\
\dot{\psi} &= [\omega_y \sin \phi + \omega_z \cos \phi] \sec \theta
\end{align*}
\]
• Need to watch for singularities at $|\theta| = \pm 90^\circ$

• If we limit

\[
\begin{align*}
0 & \leq \psi \leq 2\pi \\
-\frac{\pi}{2} & \leq \theta \leq \frac{\pi}{2} \\
0 & \leq \phi < 2\pi
\end{align*}
\]

then any possible orientation of the body can be obtained by performing the appropriate rotations in the order given.

• These are a pretty standard set of Euler angles
Quaternions

- Theorem by Euler states that any given sequence of rotations can be represented as a single rotation about a single fixed axis
- Quaterions provide a convenient parameterization of this effective axis and the rotation angle

\[
\vec{b} = \begin{bmatrix}
    b_1 \\
    b_2 \\
    b_3 \\
    b_4
\end{bmatrix} = \begin{bmatrix}
    E \sin \zeta / 2 \\
    \cos \zeta / 2
\end{bmatrix}
\]

where \(E\) is a unit vector and \(\zeta\) is a positive rotation about \(\vec{E}\)

- Notes:
  - \(||\vec{b}|| = 1\) and thus there are only 3 degrees of freedom in this formulation as well
  - If \(\vec{b}\) represents the rotational transformation from the reference frame \(a\) to reference frame \(b\), the frame \(a\) is aligned with frame \(b\) when frame \(a\) is rotated by \(\zeta\) radians about \(\vec{E}\)

- In terms of the Euler Angles:

\[
\sin \theta = -2(b_2 b_4 + b_1 b_3) \\
\phi = \arctan 2 \left[2(b_2 b_3 - b_1 b_4), 1 - 2(b_1^2 + b_2^2)\right] \\
\psi = \arctan 2 \left[2(b_1 b_2 - b_3 b_4), 1 - 2(b_2^2 + b_3^2)\right]
\]

- Pros:
  - Singularity free; Computationally efficient to do state propagation in time compared to Euler Angles

- Cons:
  - Far less intuitive - less appealing

-Refs: Kuipers, Quaternions and rotation sequences, 1999 Princeton University Press.