16.333: Lecture # 7

Approximate Longitudinal Dynamics Models

- A couple more stability derivatives
- Given mode shapes found identify simpler models that capture the main responses
More Stability Derivatives

- Recall from 6–2 that the derivative stability derivative terms $Z_{\dot{w}}$ and $M_{\dot{w}}$ ended up on the LHS as modifications to the normal mass and inertia terms
  - These are the *apparent mass* effects – some of the surrounding displaced air is “entrained” and moves with the aircraft
  - Acceleration derivatives quantify this effect
  - Significant for blimps, less so for aircraft.

- Main effect: rate of change of the normal velocity $\dot{w}$ causes a transient in the downwash $\epsilon$ from the wing that creates a change in the angle of attack of the tail some time later – *Downwash Lag* effect

- If aircraft flying at $U_0$, will take approximately $\Delta t = l_t/U_0$ to reach the tail.
  - Instantaneous downwash at the tail $\epsilon(t)$ is due to the wing $\alpha$ at time $t - \Delta t$.
    $\epsilon(t) = \frac{\partial \epsilon}{\partial \alpha} \alpha(t - \Delta t)$
  - Taylor series expansion
    $\alpha(t - \Delta t) \approx \alpha(t) - \dot{\alpha} \Delta t$
  - Note that $\Delta \epsilon(t) = -\Delta \alpha_t$. Change in the tail AOA can be computed as
    $\Delta \epsilon(t) = -\frac{d \epsilon}{d \alpha} \dot{\alpha} \Delta t = -\frac{d \epsilon}{d \alpha} \dot{l}_t \frac{l_t}{U_0} = -\Delta \alpha_t$
• For the tail, we have that the lift increment due to the change in downwash is
\[
\Delta C_L_t = C_{L_{\alpha t}} \Delta \alpha_t = C_{L_{\alpha t}} \frac{d \epsilon}{d \alpha} \frac{l_t}{U_0}
\]
The change in lift force is then
\[
\Delta L_t = \frac{1}{2} \rho (U_0^2) t S_t \Delta C_L_t
\]
• In terms of the $Z$-force coefficient
\[
\Delta C_Z = -\frac{\Delta L_t}{\frac{1}{2} \rho U_0^2 S} = -\eta \frac{S_t}{S} \Delta C_L_t = -\eta \frac{S_t}{S} C_{L_{\alpha t}} \frac{d \epsilon}{d \alpha} \frac{l_t}{U_0}
\]

• We use $\bar{c}/(2U_0)$ to nondimensionalize time, so the appropriate stability coefficient form is (note use $C_z$ to be general, but we are looking at $\Delta C_z$ from before):
\[
C_{Z_{\dot{\alpha}}} = \left( \frac{\partial C_Z}{\partial (\dot{\alpha} \bar{c}/2U_0)} \right)_0 = \frac{2U_0}{\bar{c}} \left( \frac{\partial C_Z}{\partial \dot{\alpha}} \right)_0
= -\eta \frac{2U_0 S_t}{\bar{c} S U_0} C_{L_{\alpha t}} \frac{d \epsilon}{d \alpha}
= -2\eta V_H C_{L_{\alpha t}} \frac{d \epsilon}{d \alpha}
\]

• The pitching moment due to the lift increment is
\[
\Delta M_{cg} = -l_t \Delta L_t
\Rightarrow \Delta C_{M_{cg}} = -l_t \frac{\frac{1}{2} \rho (U_0^2) t S_t \Delta C_L_t}{\frac{1}{2} \rho U_0^2 S \bar{c}}
\]
\[
= -\eta V_H \Delta C_L_t = -\eta V_H C_{L_{\alpha t}} \dot{\alpha} \frac{d \epsilon}{d \alpha} \frac{l_t}{U_0}
\]
• So that

\[ C_{M\dot{\alpha}} = \left( \frac{\partial C_M}{\partial (\dot{\alpha}c/2U_0)} \right)_0 = \frac{2U_0}{c} \left( \frac{\partial C_M}{\partial \dot{\alpha}} \right)_0 = -\eta V_H C_{L_{\alpha t}} \frac{d\epsilon}{d\alpha} \frac{l_t}{U_0} \frac{2U_0}{c} \]

\[ = -2\eta V_H C_{L_{\alpha t}} \frac{d\epsilon}{d\alpha} \frac{l_t}{c} \]

\[ \equiv \frac{l_t}{c} C_{Z\dot{\alpha}} \]

• Similarly, pitching motion of the aircraft changes the AOA of the tail. Nose pitch up at rate \( q \), increases apparent downwards velocity of tail by \( ql_t \), changing the AOA by

\[ \Delta \alpha_t = \frac{ql_t}{U_0} \]

which changes the lift at the tail (and the moment about the cg).

• Following same analysis as above: Lift increment

\[ \Delta L_t = C_{L_{\alpha t}} \frac{ql_t}{U_0} \frac{1}{2} \rho (U_0^2) t S_t \]

\[ \Delta C_Z = -\frac{1}{2} \rho (U_0^2) S = -\eta S t C_{L_{\alpha t}} \frac{ql_t}{U_0} \]

\[ C_{Zq} \equiv \left( \frac{\partial C_Z}{\partial (q\dot{c}/2U_0)} \right)_0 = \frac{2U_0}{c} \left( \frac{\partial C_Z}{\partial q} \right)_0 = -\eta \frac{2U_0}{c} \frac{l_t}{U_0} \frac{S_t}{S} C_{L_{\alpha t}} \]

\[ = -2\eta V_H C_{L_{\alpha t}} \]

• Can also show that

\[ C_{Mq} = C_{Zq} \frac{l_t}{c} \]
Approximate Aircraft Dynamic Models

- It is often good to develop simpler models of the full set of aircraft dynamics.
  
  - Provides insights on the role of the aerodynamic parameters on the frequency and damping of the two modes.
  
  - Useful for the control design work as well

- Basic approach is to recognize that the modes have very separate sets of states that participate in the response.
  
  - Short Period – primarily $\theta$ and $w$ in the same phase. The $u$ and $q$ response is very small.
  
  - Phugoid – primarily $\theta$ and $u$, and $\theta$ lags by about $90^\circ$. The $w$ and $q$ response is very small.

- Full equations from before:

$$
\begin{bmatrix}
\dot{u} \\
\dot{w} \\
\dot{q} \\
\dot{\theta}
\end{bmatrix}
= 
\begin{bmatrix}
\frac{X_u}{m} & \frac{X_w}{m} & 0 & -g \cos \Theta_0 \\
\frac{m-Z_w}{m-Z_u} & \frac{m-Z_w}{m-Z_u} & \frac{m-Z_w}{m+(Z_q+mU_0)} & -mg \sin \Theta_0 \\
I_{yy} & I_{yy} & I_{yy} - g \cos \Theta_0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
u \\
w \\
q \\
\theta
\end{bmatrix}
+ 
\begin{bmatrix}
\Delta X^c \\
\Delta Z^c \\
\Delta M^c \\
0
\end{bmatrix}
$$
• For the **Short Period** approximation,

1. Since \( u \approx 0 \) in this mode, then \( \dot{u} \approx 0 \) and can eliminate the \( X \)-force equation.

\[
\begin{bmatrix}
\dot{w} \\
\dot{q} \\
\dot{\theta}
\end{bmatrix}
= 
\begin{bmatrix}
\frac{Z_w}{m-Z_w} & \frac{Z_q+mU_0}{m-Z_w} & -mg \sin \Theta_0 \\
\frac{I_{yy}}{M_w+Z_w \Gamma} & \frac{I_{yy}}{M_q+(Z_q+mU_0) \Gamma} & -mg \sin \Theta_0 \Gamma \\
0 & \frac{I_{yy}}{m-Z_w} & 0
\end{bmatrix}
\begin{bmatrix}
w \\
q \\
\theta
\end{bmatrix}
+ 
\begin{bmatrix}
\Delta Z_c \\
\Delta M_c \\
0
\end{bmatrix}
\]

2. Typically find that \( Z_w \ll m \) and \( Z_q \ll mU_0 \). Check for 747:
   - \( Z_w = 1909 \ll m = 2.8866 \times 10^5 \)
   - \( Z_q = 4.5 \times 10^5 \ll mU_0 = 6.8 \times 10^7 \)

\[
\Gamma = \frac{M_{\dot{w}}}{m-Z_w} \Rightarrow \Gamma \approx \frac{M_{\dot{w}}}{m}
\]

\[
\begin{bmatrix}
\dot{w} \\
\dot{q} \\
\dot{\theta}
\end{bmatrix}
= 
\begin{bmatrix}
\frac{Z_w}{m-Z_w} & \frac{U_0}{M_q+(mU_0) \frac{M_{\dot{w}}}{m}} & -g \sin \Theta_0 \\
\frac{I_{yy}}{M_w+Z_w \frac{M_{\dot{w}}}{m}} & \frac{I_{yy}}{M_q+(mU_0) \frac{M_{\dot{w}}}{m}} & -mg \sin \Theta_0 \frac{M_{\dot{w}}}{m} \\
0 & \frac{I_{yy}}{m-Z_w} & 0
\end{bmatrix}
\begin{bmatrix}
w \\
q \\
\theta
\end{bmatrix}
+ 
\begin{bmatrix}
\Delta Z_c \\
\Delta M_c \\
0
\end{bmatrix}
\]

3. Set \( \Theta_0 = 0 \) and remove \( \theta \) from the model (it can be derived from \( q \))

• With these approximations, the longitudinal dynamics reduce to

\[
\dot{x}_{sp} = A_{sp} x_{sp} + B_{sp} \delta_e
\]

where \( \delta_e \) is the elevator input, and

\[
x_{sp} = \begin{bmatrix} w \\ q \end{bmatrix}, \quad A_{sp} = \begin{bmatrix} \frac{Z_w}{m} & U_0 \\ \frac{I_{yy}^{-1} (M_w + M_w Z_w / m)} & \frac{I_{yy}^{-1} (M_q + M_w U_0)} \end{bmatrix} \\
B_{sp} = \begin{bmatrix} \frac{Z_{\delta_e}}{m} \\ \frac{I_{yy}^{-1} (M_{\delta_e} + M_w Z_{\delta_e} / m)} \end{bmatrix}
\]
• Characteristic equation for this system: \( s^2 + 2ζ_{sp}ω_{sp}s + ω_{sp}^2 = 0 \), where the full approximation gives:

\[
2ζ_{sp}ω_{sp} = -\left( \frac{Z_w}{m} + \frac{M_q}{I_{yy}} + \frac{M_w}{I_{yy}}U_0 \right) \\
ω_{sp}^2 = \frac{Z_w M_q}{m I_{yy}} - \frac{U_0 M_w}{I_{yy}}
\]

• Given approximate magnitude of the derivatives for a typical aircraft, can develop a coarse approximate:

\[
2ζ_{sp}ω_{sp} ≈ -\frac{M_q}{I_{yy}} \quad \quad \quad ζ_{sp} ≈ -\frac{M_q}{2} \sqrt{\frac{-1}{U_0 M_w I_{yy}}} \\
ω_{sp}^2 ≈ -\frac{U_0 M_w}{I_{yy}} \quad \quad \quad ω_{sp} ≈ \sqrt{-\frac{U_0 M_w}{I_{yy}}}
\]

• Numerical values for 747

<table>
<thead>
<tr>
<th></th>
<th>Frequency rad/sec</th>
<th>Damping</th>
</tr>
</thead>
<tbody>
<tr>
<td>Full model</td>
<td>0.962</td>
<td>0.387</td>
</tr>
<tr>
<td>Full Approximate</td>
<td>0.963</td>
<td>0.385</td>
</tr>
<tr>
<td>Coarse Approximate</td>
<td>0.906</td>
<td>0.187</td>
</tr>
</tbody>
</table>

Both approximations give the frequency well, but full approximation gives a much better damping estimate

• Approximations showed that short period mode frequency is determined by \( M_w \) – measure of the aerodynamic stiffness in pitch.

– Sign of \( M_w \) negative if cg sufficient far forward – changes sign (mode goes unstable) when cg at the stick fixed neutral point. Follows from discussion of \( C_{Ma} \) (see 2–11)
For the Phugoid approximation, start again with:

$$\begin{bmatrix}
\dot{u} \\
\dot{w} \\
\dot{q} \\
\dot{\theta}
\end{bmatrix} = \begin{bmatrix}
\frac{X_u}{m} & \frac{X_w}{m} & 0 & -g \cos \Theta_0 \\
\frac{Z_u}{m-Z_w} & \frac{Z_w}{m-Z_w} & \frac{Z_q+mU_0}{m-Z_w} & -mg \sin \Theta_0 \\
\frac{[M_u+Z_u \Gamma]}{I_{yy}} & \frac{[M_w+Z_w \Gamma]}{I_{yy}} & \frac{[M_q+(Z_q+mU_0) \Gamma]}{I_{yy}} & 0 \\
0 & 0 & \frac{1}{I_{yy}} & 0
\end{bmatrix} \begin{bmatrix}
u \\
w \\
q \\
\theta
\end{bmatrix} + \begin{bmatrix}
\Delta X^c \\
\Delta Z^c \\
\Delta M^c
\end{bmatrix}$$

1. Changes to $w$ and $q$ are very small compared to $u$, so we can

- Set $\dot{w} \approx 0$ and $\dot{q} \approx 0$
- Set $\Theta_0 = 0$

$$\begin{bmatrix}
\dot{u} \\
0 \\
0 \\
\dot{\theta}
\end{bmatrix} = \begin{bmatrix}
\frac{X_u}{m} & \frac{X_w}{m} & 0 & -g \\
\frac{Z_u}{m-Z_w} & \frac{Z_w}{m-Z_w} & \frac{Z_q+mU_0}{m-Z_w} & 0 \\
\frac{[M_u+Z_u \Gamma]}{I_{yy}} & \frac{[M_w+Z_w \Gamma]}{I_{yy}} & \frac{[M_q+(Z_q+mU_0) \Gamma]}{I_{yy}} & 0 \\
0 & 0 & \frac{1}{I_{yy}} & 0
\end{bmatrix} \begin{bmatrix}
u \\
w \\
q \\
\theta
\end{bmatrix} + \begin{bmatrix}
\Delta X^c \\
\Delta Z^c \\
\Delta M^c
\end{bmatrix}$$

2. Use what is left of the $Z$-equation to show that with these approximations (elevator inputs)

$$\begin{bmatrix}
\frac{Z_w}{m-Z_w} & \frac{Z_q+mU_0}{m-Z_w} \\
\frac{[M_w+Z_w \Gamma]}{I_{yy}} & \frac{[M_q+(Z_q+mU_0) \Gamma]}{I_{yy}}
\end{bmatrix} \begin{bmatrix}
w \\
q
\end{bmatrix} = - \begin{bmatrix}
\frac{Z_u}{m-Z_w} \\
\frac{[M_u+Z_u \Gamma]}{I_{yy}}
\end{bmatrix} \begin{bmatrix}
u - \frac{Z_{\delta e}}{m-Z_w} \\
\frac{[M_{\delta e}+Z_{\delta e} \Gamma]}{I_{yy}}
\end{bmatrix} \delta_e$$

3. Use ($Z_{\dot{w}} \ll m$ so $\Gamma \approx \frac{M_{\dot{w}}}{m}$) and ($Z_q \ll mU_0$) so that:

$$\begin{bmatrix}
\frac{Z_w}{M_w + Z_w \frac{M_{\dot{w}}}{m}} & \frac{mU_0}{[M_q + U_0 M_{\dot{w}}]} \\
\frac{Z_u}{[M_u + Z_u \frac{M_{\dot{w}}}{m}]}
\end{bmatrix} \begin{bmatrix}
w \\
q
\end{bmatrix} = - \begin{bmatrix}
\frac{Z_{\delta e}}{M_{\delta e} + Z_{\delta e} \frac{M_{\dot{w}}}{m}}
\end{bmatrix} \begin{bmatrix}
u - \frac{Z_{\delta e}}{M_{\delta e} + Z_{\delta e} \frac{M_{\dot{w}}}{m}}
\end{bmatrix} \delta_e$$
4. Solve to show that
\[
\begin{bmatrix}
  w \\
  q
\end{bmatrix} = \begin{bmatrix}
  \frac{mU_0 M_u - Z_u M_q}{Z_w M_q - mU_0 M_w} \\
  \frac{Z_u M_w - Z_w M_u}{Z_w M_q - mU_0 M_w}
\end{bmatrix} u + \begin{bmatrix}
  \frac{mU_0 M_{\delta e} - Z_{\delta e} M_q}{Z_w M_q - mU_0 M_w} \\
  \frac{Z_{\delta e} M_w - Z_w M_{\delta e}}{Z_w M_q - mU_0 M_w}
\end{bmatrix} \delta_e
\]

5. Substitute into the reduced equations to get full approximation:
\[
\begin{bmatrix}
  \dot{u} \\
  \dot{\theta}
\end{bmatrix} = \begin{bmatrix}
  \frac{X_u}{m} + \frac{X_w}{m} \left( \frac{mU_0 M_u - Z_u M_q}{Z_w M_q - mU_0 M_w} \right) - g \\
  \left( \frac{Z_u M_w - Z_w M_u}{Z_w M_q - mU_0 M_w} \right)
\end{bmatrix} \begin{bmatrix}
  u \\
  \theta
\end{bmatrix}
+ \begin{bmatrix}
  \frac{X_{\delta e}}{m} + \frac{X_w}{m} \left( \frac{mU_0 M_{\delta e} - Z_{\delta e} M_q}{Z_w M_q - mU_0 M_w} \right) \\
  \frac{Z_{\delta e} M_w - Z_w M_{\delta e}}{Z_w M_q - mU_0 M_w}
\end{bmatrix} \delta_e
\]

6. Still a bit complicated. Typically get that
- \(|M_u Z_w| \ll |M_w Z_u|\) (1.4:4)
- \(|M_w U_0 m| \gg |M_q Z_w|\) (1:0.13)
- \(|M_u X_w / M_w| \ll X_u \) small

7. With these approximations, the longitudinal dynamics reduce to the coarse approximation
\[
\dot{x}_{ph} = A_{ph} x_{ph} + B_{ph} \delta_e
\]
where \(\delta_e\) is the elevator input.
And

$$x_{ph} = \begin{bmatrix} u \\ \theta \end{bmatrix}$$

$$A_{ph} = \begin{bmatrix} \frac{X_u}{m} & -g \\ -Z_u & 0 \end{bmatrix}$$

$$B_{ph} = \begin{bmatrix} \frac{(X_{\delta_e} - \left[ \frac{X_w}{M_w} \right] M_{\delta_e})}{m} \\ \frac{m}{mU_0} \left[ -Z_{\delta_e} + \left[ \frac{Z_w}{M_w} \right] M_{\delta_e} \right] \end{bmatrix}$$

8. Which gives

$$2\zeta_{ph}\omega_{ph} = -\frac{X_u}{m}$$

$$\omega_{ph}^2 = -\frac{gZ_u}{mU_0}$$

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**Numerical values for 747**

<table>
<thead>
<tr>
<th></th>
<th>Frequency rad/sec</th>
<th>Damping</th>
</tr>
</thead>
<tbody>
<tr>
<td>Full model</td>
<td>0.0673</td>
<td>0.0489</td>
</tr>
<tr>
<td>Full Approximate</td>
<td>0.0670</td>
<td>0.0419</td>
</tr>
<tr>
<td>Coarse Approximate</td>
<td>0.0611</td>
<td>0.0561</td>
</tr>
</tbody>
</table>
• Further insights: recall that
\[
\left( \frac{U_0}{QS} \right) \left( \frac{\partial Z}{\partial u} \right)_0 = - \left( \frac{U_0}{QS} \right) \left( \frac{\partial L}{\partial u} \right)_0 \equiv -(C_{L_u} + 2C_{L_0})
\]
\[
= - \frac{M^2}{1 - M^2} C_{L_0} - 2C_{L_0} \approx -2C_{L_0}
\]
so
\[
Z_u \equiv \left( \frac{\partial Z}{\partial u} \right)_0 = \left( \frac{\rho U_o S}{2} \right) (-2C_{L_0}) = - \frac{2mg}{U_0}
\]

• Then
\[
\omega_{ph} = \sqrt{-gZ_u \over mU_0} = \sqrt{mg^2 \over mU_0^2} = \sqrt{2 \cdot g \over U_0}
\]
which is exactly what Lanchester’s approximation gave \( \Omega \approx \sqrt{2 \cdot g \over U_0} \)

• Note that
\[
X_u \equiv \left( \frac{\partial X}{\partial u} \right)_0 = \left( \frac{\rho U_o S}{2} \right) (-2C_{D_0}) = -\rho U_o S C_{D_0}
\]
and
\[
2mg = \rho U_o^2 S C_{L_0}
\]
so
\[
\zeta_{ph} = - \frac{X_u}{2m \omega_{ph}} = - \frac{X_u U_0}{2\sqrt{2}mg}
\]
\[
= \frac{1}{\sqrt{2}} \left( \frac{\rho U_o^2 S C_{D_0}}{C_{D_0}} \right)
\]
\[
= \frac{1}{\sqrt{2}} \left( \frac{C_{D_0}}{C_{L_0}} \right)
\]
so the damping ratio of the approximate phugoid mode is inversely proportional to the lift to drag ratio.
Freq Comparison from elevator (Phugoid Model) – B747 at M=0.8. **Blue**– Full model, **Black**– Full approximate model, **Magenta**– Coarse approximate model.
Freq Comparison from elevator (Short Period Model) – B747 at M=0.8. Blue– Full model, Magenta– Approximate model
Summary

- Approximate longitudinal models are fairly accurate

- Indicate that the aircraft responses are mainly determined by these stability derivatives:

<table>
<thead>
<tr>
<th>Property</th>
<th>Stability derivative</th>
</tr>
</thead>
<tbody>
<tr>
<td>Damping of the short period</td>
<td>$M_q$</td>
</tr>
<tr>
<td>Frequency of the short period</td>
<td>$M_w$</td>
</tr>
<tr>
<td>Damping of the Phugoid</td>
<td>$X_u$</td>
</tr>
<tr>
<td>Frequency of the Phugoid</td>
<td>$Z_u$</td>
</tr>
</tbody>
</table>
- Given a change in $\alpha$, expect changes in $u$ as well. These will both impact the lift and drag of the aircraft, requiring that we re-trim throttle setting to maintain whatever aspects of the flight condition might have changed (other than the ones we wanted to change). We have:

\[
\begin{bmatrix}
\Delta L \\
\Delta D
\end{bmatrix} = \begin{bmatrix} L_u & L_\alpha \\
D_u & D_\alpha \end{bmatrix} \begin{bmatrix} u \\
\Delta \alpha \end{bmatrix}
\]

But to maintain $L = W$, want $\Delta L = 0$, so $u = -\frac{L_\alpha}{L_u} \Delta \alpha$

Giving $\Delta D = \left(-\frac{L_\alpha}{L_u} D_u + D_\alpha\right) \Delta \alpha$

\[
C_{D_\alpha} = \frac{2C_{L_0}}{\pi eAR} C_{L_\alpha} \rightarrow D_\alpha = QSC_{D_\alpha}
\]

\[
\rightarrow L_\alpha = QSC_{L_\alpha}
\]

\[
D_u = \frac{QS}{U_0} (2C_{D_0}) \quad (4 - 16)
\]

\[
L_u = \frac{QS}{U_0} (2C_{L_0}) \quad (4 - 17)
\]

\[
\Delta D = QS \left(-\frac{C_{L_\alpha}}{2C_{L_0}/U_0} \left(\frac{2C_{D_0}}{U_0}\right) + C_{D_\alpha}\right) \Delta \alpha
\]

\[
= QS \left(-C_{D_0} + \frac{2C_{L_0}^2}{\pi eAR}\right) C_{L_\alpha} \Delta \alpha
\]

\[
\tan \Delta \gamma = \frac{(T_0 + \Delta T) - (D_0 + \Delta D)}{L_0 + \Delta L} = -\frac{\Delta D}{L_0}
\]

\[
= \left(\frac{C_{D_0}}{C_{L_0}} - \frac{2C_{L_0}}{\pi eAR}\right) \frac{C_{L_\alpha}}{C_{L_0}} \Delta \alpha
\]

For 747 (Reid 165 and Nelson 416), $AR = 7.14$, so $\pi eAR \approx 18$, $C_{L_0} = 0.654$ $C_{D_0} = 0.043$, $C_{L_\alpha} = 5.5$, for $\Delta \alpha = -0.0185 \text{rad}$ (6–7) $\Delta \gamma = -0.0006 \text{rad}$. This is the opposite sign to the linear simulation results, but they are both very small numbers.