Lecture 2 The Two Body Problem Continued

The Eccentricity Vector or The Laplace Vector

$$\mu e = v \times h - \frac{\mu}{r} r$$

Explicit Form of the Velocity Vector #3.1

Using the expansion of the triple vector product $a \times (b \times c) = (a \cdot c)b - (a \cdot b)c$ we have

$$h \times \mu e = h \times (v \times h) - \frac{\mu}{r} h \times r = h^2 v - \mu h - \mu h \times i_r = h^2 v - \mu h \times i_r$$

since $h$ and $v$ are perpendicular. Therefore:

$$h \times \mu e \implies v = \frac{\mu}{h} i_h \times (e + i_r)$$

or

$$\frac{h v}{\mu} = i_h \times (e i_e + i_r) = e i_h \times i_e + i_h \times i_r = e i_p + i_\theta$$

Then since

$$i_p = \sin f i_r + \cos f i_\theta$$

we have

$$\frac{h v}{\mu} = e \sin f i_r + (1 + e \cos f) i_\theta$$

which is the basic relation for representing the velocity vector in the Hodograph Plane.

See Page 1 of Lecture 4

Conservation of Energy

$$\frac{h v}{\mu} \cdot \frac{h v}{\mu} = \frac{p}{\mu} v \cdot v = 2(1 + e \cos f) + e^2 - 1 = 2 \times \frac{p}{r} - (1 - e^2) = \frac{2}{r} - \frac{1}{a}$$

which can be written in either of two separate forms each having its own name:

**Energy Integral**

$$\frac{1}{2} v^2 - \frac{\mu}{r} = -\frac{\mu}{2a} = \frac{1}{2} c_3$$

**Vis-Viva Integral**

$$v^2 = \mu \left(\frac{2}{r} - \frac{1}{a}\right)$$

The constant $c_3$ is used by Forest Ray Moulton, a Professor at the University of Chicago in his 1902 book “An Introduction to Celestial Mechanics” — the first book on the subject written by an American.

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Conic Sections

Ellipse or Hyperbola in rectangular coordinates ($e \neq 1$)

\[ y^2 = r^2 - x^2 = (p - ex)^2 - x^2 = (1 - e^2)[a^2 - (x + ea)^2] \]

\[
\frac{(x + ea)^2}{a^2} + \frac{y^2}{a^2(1 - e^2)} = 1
\]

Semiminor Axis:

\[ b^2 = |a^2(1 - e^2)| = |a|p \]

Fig. 3.1 from *An Introduction to the Mathematics and Methods of Astrodynamics*. Courtesy of AIAA. Used with permission.

Parabola in rectangular coordinates ($e = 1$)

\[ y^2 = r^2 - x^2 = (p - x)^2 - x^2 \implies y^2 = 2p\left(\frac{1}{2}p - x\right) \]

Fig. 3.2 from *An Introduction to the Mathematics and Methods of Astrodynamics*. Courtesy of AIAA. Used with permission.

Fig. 3.3 from *An Introduction to the Mathematics and Methods of Astrodynamics*. Courtesy of AIAA. Used with permission.
Alternate Forms of the Equation of Orbit

**Origin at focus** \( r + ex = p \)

**Origin at center** \( r + ex = a \)

With \( x \) now measured from the center which is at a distance \( ae \) from the focus, then

\[
\begin{align*}
  r + ex &= p \\
  r + e(x - ae) &= p = a(1 - e^2) \\
  r + ex &= a
\end{align*}
\]

**Origin at pericenter** \( r + ex = q \)

With \( x \) now measured from pericenter which is at a distance of \( a \) from the center and a distance of \( q = a(1 - e) \) from the focus, then

\[
\begin{align*}
  r + ex &= p \\
  r + e(x + q) &= p = q(1 + e) \\
  r + ex &= q
\end{align*}
\]

These are useful to derive other properties of conic sections:

- **Focus-Directrix Property:** \( r = p - ex \): Page 144
  \[
  PF = e \left( \frac{p}{e} - x \right) = e \times PN \]
  or
  \[
  \frac{PF}{PN} = e
  \]

- **Focal-Radii Property:** \( r = a - ex \): Page 145
  \[
  PF^2 = (x - ea)^2 + y^2 \\
  PF^*^2 = (x + ea)^2 + y^2
  \]
  so that
  \[
  PF^*^2 = PF^2 + 4aex \\
  = r^2 + 4aex \\
  = (a - ex)^2 + 4aex = (a + ex)^2
  \]
  \[
  PF^* = \begin{cases} 
  a + ex & \text{ellipse} & a > 0 \\
  -(a + ex) & \text{hyperbola} & a < 0, \ x < 0
  \end{cases}
  \]
  Thus,
  \[
  \begin{align*}
  PF^* + PF &= 2a & \text{ellipse} \\
  PF^* - PF &= -2a & \text{hyperbola}
  \end{align*}
  \]

- **Euler’s Universal Form:** From \( r = q - ex \): Page 143
  \[
  y^2 = r^2 - (q + x)^2 = (q - ex)^2 - (q + x)^2
  \]
  Then
  \[
  y^2 = -(1 + e)[2qx + (1 - e)x^2]
  \]
Basic Two-Body Relations

Vector Equations of Motion
\[ \frac{d^2 \mathbf{r}}{dt^2} + \frac{\mu}{r^3} \mathbf{r} = 0 \quad \text{or} \quad \frac{d\mathbf{v}}{dt} = -\frac{\mu}{r^3} \mathbf{r} \]

Angular Momentum Vector
\[ \mathbf{r} \times \frac{d\mathbf{v}}{dt} = 0 \implies \mathbf{r} \times \mathbf{v} = \text{constant} \equiv \mathbf{h} \]

Eccentricity Vector
\[ \frac{d\mathbf{v}}{dt} \times \mathbf{h} \implies \frac{1}{\mu} \mathbf{v} \times \mathbf{h} - \mathbf{i}_r = \text{constant} \equiv \mathbf{e} \]

Equation of Orbit
\[ \mu \mathbf{e} \cdot \mathbf{r} \implies \mathbf{r} = \frac{h^2/\mu}{1 + e \cos f} = \frac{p}{1 + e \cos f} \]

Velocity Vector
\[ \mathbf{h} \times \mu \mathbf{e} \implies \mathbf{v} = \frac{1}{p} \mathbf{h} \times (\mathbf{e} + \mathbf{i}_r) \]

Orbital Parameter \[ p \]

Dynamics Definition: \( p \equiv \frac{h^2}{\mu} \)
Geometric Definition: \( p = a(1 - e^2) \)

Total Energy or Semimajor Axis or Mean Distance \[ a \]

Dynamics Definition: \( \frac{1}{2} v^2 - \frac{\mu}{r} = \text{constant} \equiv -\frac{\mu}{2a} \)
Geometric Definition: \( \frac{(x + ea)^2}{a^2} + \frac{y^2}{a^2(1 - e^2)} = 1 \)

Eqs. of Motion in Polar Coord.
\[ \frac{d^2 r}{dt^2} - r \left( \frac{d\theta}{dt} \right)^2 + \frac{\mu}{r^2} = 0 \quad \frac{d}{dt} \left( r^2 \frac{d\theta}{dt} \right) = 0 \]

Kepler’s Laws

Second Law
\[ \frac{dA}{dt} = \frac{1}{2} r^2 \frac{d\theta}{dt} = \text{constant} = \frac{h}{2} \]

First Law
\[ r = \frac{p}{1 + e \cos f} \quad \text{or} \quad r = p - ex \]

Third Law
\[ \frac{\pi ab}{P^2} = \frac{h}{2} \implies \frac{a^3}{P^2} = \text{constant} = \frac{\mu}{4\pi^2} \]

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