16.346 Astrodynamics
Fall 2008

Lecture 4 The Initial-Value Problem

Polar Coordinates and Orbital Plane Coordinates

From Lecture 2

$$\begin{bmatrix} \frac{1}{r}\mathbf{r} \\ \frac{h}{\mu}\mathbf{v} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ e\sin f & 1+e\cos f \end{bmatrix} \begin{bmatrix} \mathbf{i}_r \\ \mathbf{i}_\theta \end{bmatrix} \qquad \begin{bmatrix} \mathbf{i}_r \\ \mathbf{i}_\theta \end{bmatrix} = \begin{bmatrix} \cos f & \sin f \\ -\sin f & \cos f \end{bmatrix} \begin{bmatrix} \mathbf{i}_e \\ \mathbf{i}_p \end{bmatrix}$$

Representation in the Hodograph Plane

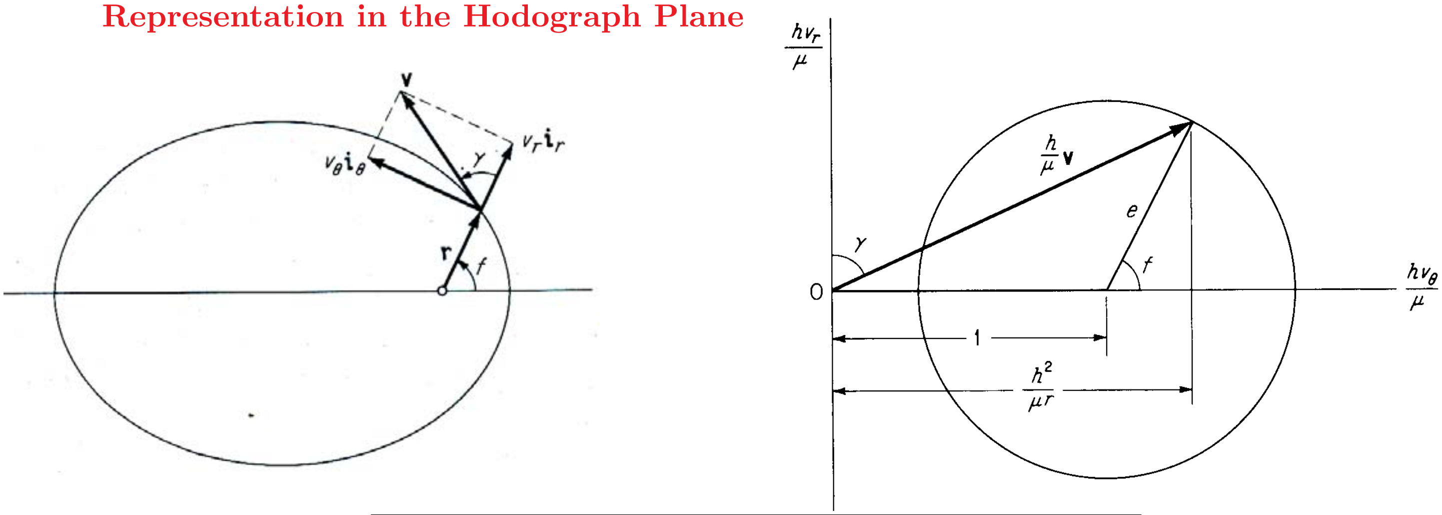

Fig. 3.5 (a) and (b) from *An Introduction to the Mathematics and Methods of Astrodynamics.* Courtesy of AIAA. Used with permission.

Physical Plane **Hodograph Plane**

Flight-Direction Angle

From the position and velocity equations in polar coordinates at the top of this page:

$$\mathbf{r}\cdot\mathbf{v} = \frac{\mu r e \sin f}{h}$$

In terms of the flight-direction angle γ (shown in the figures above),

$$\begin{aligned} \mathbf{r} &= r\,\mathbf{i}_r \\ \mathbf{v} &= v\cos\gamma\,\mathbf{i}_r + v\sin\gamma\,\mathbf{i}_\theta \end{aligned} \qquad \Longrightarrow \qquad \mathbf{r}\cdot\mathbf{v} = rv\cos\gamma$$

Also

$$h = |\mathbf{r}\times\mathbf{v}| = rv\sin\gamma$$

Therefore:

$$\boxed{\sigma \overset{\text{def}}{=} \frac{\mathbf{r}\cdot\mathbf{v}}{\sqrt{\mu}}} = \frac{rv\cos\gamma}{\sqrt{\mu}} = \frac{h\cot\gamma}{\sqrt{\mu}} \quad \text{or, since} \quad p = \frac{h^2}{\mu} \quad \text{we have}$$

$$\boxed{\sigma = \sqrt{p}\cot\gamma}$$

$$\begin{bmatrix} \mathbf{r} \\ \mathbf{v} \end{bmatrix} = \begin{bmatrix} r\cos f & r\sin f \\ -\frac{\mu}{h}\sin f & \frac{\mu}{h}(e+\cos f) \end{bmatrix} \begin{bmatrix} \mathbf{i}_e \\ \mathbf{i}_p \end{bmatrix} \qquad \begin{bmatrix} \mathbf{i}_e \\ \mathbf{i}_p \end{bmatrix} = \begin{bmatrix} \frac{\mu}{h^2}(e+\cos f_0) & -\frac{r_0}{h}\sin f_0 \\ \frac{\mu}{h^2}\sin f_0 & \frac{r_0}{h}\cos f_0 \end{bmatrix} \begin{bmatrix} \mathbf{r}_0 \\ \mathbf{v}_0 \end{bmatrix}$$

$$\boxed{\begin{aligned} \mathbf{r} &= F\,\mathbf{r}_0 + G\,\mathbf{v}_0 \\ \mathbf{v} &= F_t\,\mathbf{r}_0 + G_t\,\mathbf{v}_0 \end{aligned}} \quad \text{or} \quad \begin{bmatrix} \mathbf{r} \\ \mathbf{v} \end{bmatrix} = \mathbf{\Phi} \begin{bmatrix} \mathbf{r}_0 \\ \mathbf{v}_0 \end{bmatrix} \quad \text{with} \quad \mathbf{\Phi} = \begin{bmatrix} F & G \\ F_t & G_t \end{bmatrix} \quad \text{and} \quad |\mathbf{\Phi}| = 1$$

The value of the determinant $|\mathbf{\Phi}| = FG_t - GF_t = 1$ follows from the conservation of angular momentum

$$\mathbf{r}\times\mathbf{v} = (FG_t - GF_t)\,\mathbf{r}_0\times\mathbf{v}_0 = \mathbf{r}_0\times\mathbf{v}_0$$

Lagrange Coefficients in Terms of the True Anomaly Difference

$$\begin{aligned} F &= 1 - \frac{r}{p}(1-\cos\theta) & G &= \frac{rr_0}{\sqrt{\mu p}}\sin\theta \\ F_t &= \frac{\sqrt{\mu}}{r_0 p}\left[\sigma_0(1-\cos\theta) - \sqrt{p}\sin\theta\right] & G_t &= 1 - \frac{r_0}{p}(1-\cos\theta) \end{aligned} \qquad (3.42)$$

where $\dfrac{r}{r_0} = \dfrac{p}{r_0 + (p - r_0)\cos\theta - \sqrt{p}\,\sigma_0\sin\theta}$ with $\theta = f - f_0$ and $\sigma_0 = \dfrac{\mathbf{r}_0\cdot\mathbf{v}_0}{\sqrt{\mu}}$

Lagrange Coefficients in Terms of the Eccentric Anomaly Difference Page 162

Define $\varphi = E - E_0$. Then:

$$\begin{aligned} F &= 1 - \frac{a}{r_0}(1-\cos\varphi) & \sqrt{\mu}\,G &= a\sigma_0(1-\cos\varphi) + r_0\sqrt{a}\sin\varphi \\ F_t &= -\frac{\sqrt{\mu a}}{rr_0}\sin\varphi & G_t &= 1 - \frac{a}{r}(1-\cos\varphi) \end{aligned} \qquad (4.41)$$

where $r = a + (r_0 - a)\cos\varphi + \sigma_0\sqrt{a}\sin\varphi$ and $\sigma_0 = \dfrac{\mathbf{r}_0\cdot\mathbf{v}_0}{\sqrt{\mu}} \equiv \sqrt{p}\cot\gamma_0$

Kepler's Equation is then

$$M - M_0 = \sqrt{\frac{\mu}{a^3}}\,(t - t_0) = (E - e\sin E) - (E_0 - e\sin E_0)$$

or, in terms of $\varphi = E - E_0$

$$\sqrt{\frac{\mu}{a^3}}\,(t - t_0) = \varphi + \frac{\sigma_0}{\sqrt{a}}(1-\cos\varphi) - \left(1 - \frac{r_0}{a}\right)\sin\varphi \qquad (4.43)$$

Since $\tan\frac{1}{2}f$ is obtained directly as the solution of Barker's equation, it is more convenient to express all trigonometric functions in terms of this function of the true anomaly.

Thus, the position and velocity vectors for the parabola in orbital plane coordinates are

$$\begin{aligned}\mathbf{r} &= \frac{p}{2}(1-\tan^2\tfrac{1}{2}f)\,\mathbf{i}_e + p\tan\tfrac{1}{2}f\,\mathbf{i}_p \\ \mathbf{v} &= -\frac{\sqrt{\mu p}}{r}\tan\tfrac{1}{2}f\,\mathbf{i}_e + \frac{\sqrt{\mu p}}{r}\,\mathbf{i}_p\end{aligned}$$

Define
$$\sigma = \frac{\mathbf{r}\cdot\mathbf{v}}{\sqrt{\mu}} = \sqrt{p}\tan\tfrac{1}{2}f = \sqrt{p}\cot\gamma$$

Therefore, with $\chi = \sigma - \sigma_0$, we have

$$F = 1 - \frac{\chi^2}{2r_0} \qquad G = \frac{\chi}{2\sqrt{\mu}}(2r_0 + \sigma_0\chi)$$
$$F_t = -\frac{\sqrt{\mu}\,\chi}{rr_0} \qquad G_t = 1 - \frac{\chi^2}{2r}$$

together with Barker's equation and the equation of orbit:

$$\begin{aligned}6\sqrt{\mu}(t-t_0) &= 6r_0\chi + 3\sigma_0\chi^2 + \chi^3 \\ r &= r_0 + \sigma_0\chi + \tfrac{1}{2}\chi^2 \\ \sigma &= \sigma_0 + \chi\end{aligned}$$

Solving the Generalized Form of Barker's Equation

The solution is

$$\chi = \sqrt{p}\,z - \sigma_0$$

where z is obtained by solving the cubic equation

$$z^3 + 3z = 2B$$

with

$$B = \frac{1}{p^{\frac{3}{2}}}\left[\sigma_0(r_0 + p) + 3\sqrt{\mu}(t-t_0)\right]$$

Therefore, all the solution methods developed for Barker's equation are applicable without modification provided that $B > 0$.

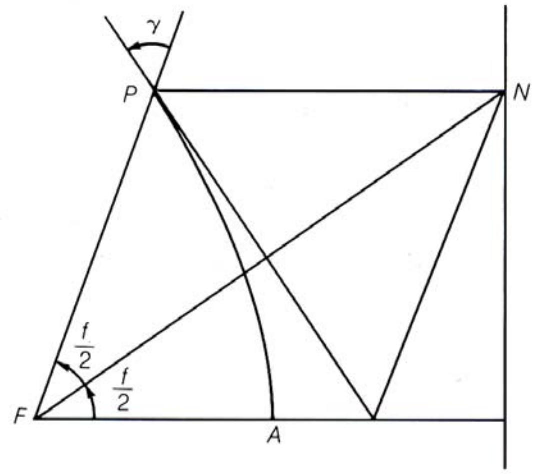

Fig. 4.7 from *An Introduction to the Mathematics and Methods of Astrodynamics.* Courtesy of AIAA. Used with permission.