Orbital Parameter from the Semimajor Axis

If \( \mathbf{r}_1 \times \mathbf{r}_2 \neq \mathbf{0} \), write
\[
\mathbf{e} = A \mathbf{i}_{\mathbf{r}_1} + B \mathbf{i}_{\mathbf{r}_2}
\]

Then, from the equation of orbit \( \mathbf{r} = p - \mathbf{e} \cdot \mathbf{r} \) obtain
\[
\begin{align*}
\mathbf{e} \cdot \mathbf{i}_{\mathbf{r}_1} &= \frac{p}{r_1} - 1 = A + B \cos \theta & \Rightarrow A \sin^2 \theta &= \left( \frac{p}{r_1} - 1 \right) - \left( \frac{p}{r_2} - 1 \right) \cos \theta \\
\mathbf{e} \cdot \mathbf{i}_{\mathbf{r}_2} &= \frac{p}{r_2} - 1 = A \cos \theta + B & \Rightarrow B \sin^2 \theta &= \left( \frac{p}{r_2} - 1 \right) - \left( \frac{p}{r_1} - 1 \right) \cos \theta
\end{align*}
\]

Next
\[
\mathbf{e} \cdot \mathbf{e} = e^2 = 1 - \frac{p}{a} = A^2 + 2AB \cos \theta + B^2
\]

so that
\[
\left( \frac{p}{r_1} - 1 \right)^2 - 2 \left( \frac{p}{r_1} - 1 \right) \left( \frac{p}{r_2} - 1 \right) \cos \theta + \left( \frac{p}{r_2} - 1 \right)^2 = \left( 1 - \frac{p}{a} \right) \sin^2 \theta
\]

or, after simplification (using the trig formulas for the semiperimeter on Page 4),
\[
\left( \frac{p}{p_m} \right)^2 - 2D \frac{p}{p_m} + 1 = 0 \quad \text{where} \quad D = \frac{r_1 + r_2}{c} - \frac{r_1r_2}{ac} \cos^2 \frac{1}{2} \theta = \frac{r_1 + r_2}{c} - \frac{s(s - c)}{ac}
\]

Therefore:
\[
\frac{p}{p_m} = D \pm \sqrt{D^2 - 1}
\]

Semimajor Axis from the Parameter

Alternately, from the quadratic equation, \( 1/a \) can be determined from \( p/p_m \) using
\[
\frac{r_1 + r_2}{c} - \frac{s(s - c)}{ac} = \frac{D}{2} = \frac{1}{2} \left( \frac{p_m}{p} + \frac{p}{p_m} \right) \quad \Rightarrow \quad \frac{1}{a} = \frac{r_1 + r_2 - cD}{s(s - c)}
\]

Semimajor Axis of the Minimum-Energy Orbit

Since the parameter of the minimum-energy orbit is \( p = p_m \), then \( a_m \) can be determined from the last boxed equation (with \( p/p_m = 1 \)). This equation becomes
\[
D = 1 \quad \text{or} \quad a_m (r_1 + r_2) - r_1r_2 \cos^2 \frac{1}{2} \theta = a_m c
\]

Introduce the semiperimeter of the triangle \( s = \frac{1}{2} (r_1 + r_2 + c) \) and use one of the equations from Problem G–4 in Appendix G of the textbook to write
\[
a_m (r_1 + r_2 - c) = r_1r_2 \cos^2 \frac{1}{2} \theta = s(s - c) \quad \text{or} \quad 2a_m (s - c) = s(s - c)
\]

Hence
\[
a_m = \frac{1}{2} s = \frac{1}{4} (r_1 + r_2 + c)
\]
Orbit Tangents and the Transfer-Angle Bisector  

The line connecting the focus and the point of intersection of the orbital tangents at the terminals bisects the transfer angle.

\[
\sqrt{r_1 r_2} = \begin{cases} 
FN \cos \frac{1}{2}(E_2 - E_1) & \text{ellipse} \\
FN & \text{parabola} \\
FN \cosh \frac{1}{2}(H_2 - H_1) & \text{hyperbola}
\end{cases}
\]

Locus of the Eccentricity Vectors of the Boundary-Value Problem  

Equation of orbit \( e \cdot r = p - r \) at \( P_1 \) and \( P_2 \):

\[
e \cdot r_1 = p - r_1 \quad \Rightarrow \quad e \cdot (r_2 - r_1) = r_1 - r_2 \quad \text{or} \quad -e \cdot i_c = \frac{r_2 - r_1}{c}
\]

Fig. 6.7 from *An Introduction to the Mathematics and Methods of Astrodynamics*. Courtesy of AIAA. Used with permission.

Locus of the Eccentricity Vectors of the Boundary-Value Problem  

Equation of orbit \( e \cdot r = p - r \) at \( P_1 \) and \( P_2 \):

\[
e \cdot r_1 = p - r_1 \quad \Rightarrow \quad e \cdot (r_2 - r_1) = r_1 - r_2 \quad \text{or} \quad -e \cdot i_c = \frac{r_2 - r_1}{c}
\]

Fig. 6.11 from *An Introduction to the Mathematics and Methods of Astrodynamics*. Courtesy of AIAA. Used with permission.
Locus of the Vacant Focus

Elliptic Orbits:
\[
\begin{align*}
    r_1 + P_1 F^* &= 2a \\
    r_2 + P_2 F^* &= 2a \\
\end{align*}
\]
\[\Rightarrow P_2 F^* - P_1 F^* = -(r_2 - r_1) = 2a^* \]

Note: \(2a \geq r_1 + r_2 + c = 2s = 2a_m\)

Hyperbolic Orbits:
\[
\begin{align*}
    r_1 - P_1 F^* &= 2a \\
    r_2 - P_2 F^* &= 2a \\
\end{align*}
\]
\[\Rightarrow P_1 F^* - P_2 F^* = -(r_2 - r_1) = 2a^* \]

Note: \(F_0^*\): Rectilinear orbit from \(P_1\) to \(P_2\) with \(a = 0\) and \(e = \infty\)

Note: \(\tilde{F}_0^*\): Two straight-line segments \(P_2\) to \(F\) to \(P_1\) with \(a = 0\) and \(e = \sec \frac{1}{2} \theta\)

Hyperbolic locus of vacant foci:
\[
2a^* = -(r_2 - r_1) \quad e^* = \frac{c}{r_2 - r_1} \]

Fig. 6.17 and 6.18 from An Introduction to the Mathematics and Methods of Astrodynamics. Courtesy of AIAA. Used with permission.
The Semiperimeter of a Triangle

One of the twelve greatest theorems of all times, according to the author William Dunham in his book *Journey through Genius* published by John Wiley & Sons, Inc., was the formula for the area of a triangle involving only the lengths of the three sides which was discovered by either Archimedes (287 B.C. – 212 B.C.) or a centuary later, by Heron.

\[
\text{Area of a Triangle} = \sqrt{s(s - a)(s - b)(s - c)}
\]

where

\[
s = \frac{1}{2}(a + b + c)
\]

There are other remarkable formulas given in Appendix G on Page 364 of our textbook, namely, Problems G–6 and G–7.

Several trigonometric formulas given in Problem G-5 are useful for our work with the Boundary-Value Problem. In particular

\[
\sin \frac{1}{2} \alpha = \sqrt{\frac{(s - b)(s - c)}{bc}} \quad \text{and} \quad \cos \frac{1}{2} \alpha = \sqrt{\frac{s(s - a)}{bc}}
\]

Indeed, you might find it instructive to derive these and the equation

\[
\sin \alpha = \frac{2}{bc} \sqrt{s(s - a)(s - b)(s - c)}
\]

The angle \( \alpha \) is the angle opposite side \( a \) and between the sides \( b \) and \( c \).