Lecture 14  Hypergeometric Functions and Continued Fractions

John Wallis’ Hypergeometric Series
\[ a + a(a + b) + a(a + b)(a + 2b) + \cdots + a(a + b)(a + 2b) \cdots [a + (n - 1)b] + \cdots \]

Hypergeometric Function  Named by Gauss’ mentor Johann Pfaff 1765–1825

In the year 1812, Carl Friedrich Gauss published his book entitled:

\[
\text{GENERAL INVESTIGATIONS CONCERNING THE INFINITE SERIES}
\]

\[
1 + \frac{\alpha\beta}{1 \cdot \gamma} x + \frac{\alpha(\alpha + 1)\beta(\beta + 1)}{1 \cdot 2 \cdot \gamma(\gamma + 1)} xx + \frac{\alpha(\alpha + 1)(\alpha + 2)\beta(\beta + 1)(\beta + 2)}{1 \cdot 2 \cdot 3 \cdot \gamma(\gamma + 1)(\gamma + 2)} x^3 + \text{etc.}
\]

We use the symbol \( F(\alpha, \beta; \gamma; x) \) to represent this series.

Examples of Hypergeometric Functions

\[
\log(1 + x) = xF(1, 1; 2; -x) \quad \sin x = \lim_{\beta \to \infty} xF\left(\alpha; \frac{3}{2}; -\frac{x^2}{4\alpha\beta}\right)
\]

\[
\arctan x = xF\left(\frac{1}{2}, 1; \frac{3}{2}; -x^2\right) \quad \cos x = \lim_{\beta \to \infty} F\left(\alpha; \frac{1}{2}; -\frac{x^2}{4 \alpha\beta}\right)
\]

Gauss’ Differential Equation

\[
x(1-x) \frac{d^2y}{dx^2} + [\gamma - (\alpha + \beta + 1)x] \frac{dy}{dx} - \alpha\beta y = 0
\]

has the general solution

\[ y = c_1 F(\alpha, \beta; \gamma; x) + c_2 x^{1-\gamma} F(\alpha - \gamma + 1, \beta - \gamma + 1; 2 - \gamma; x) \]

Gauss’ Continued Fraction Expansion

\[
F_0 = F(\alpha, \beta; \gamma; x)
\]

\[
F_1 = F(\alpha, \beta + 1; \gamma + 1; x) \quad F_1 - F_0 = \delta_1 x F_2 \quad \delta_1 = \frac{\alpha(\gamma - \beta)}{\gamma(\gamma + 1)}
\]

\[
F_2 = F(\alpha + 1, \beta + 1; \gamma + 2; x) \quad F_2 - F_1 = \delta_2 x F_3 \quad \delta_2 = \frac{(\beta + 1)(\gamma - \alpha + 1)}{(\gamma + 1)(\gamma + 2)}
\]

\[
F_3 = F(\alpha + 1, \beta + 2; \gamma + 3; x) \quad F_3 - F_2 = \delta_3 x F_4 \quad \delta_3 = \frac{(\alpha + 1)(\gamma - \beta + 1)}{(\gamma + 2)(\gamma + 3)}
\]

\[
F_4 = F(\alpha + 2, \beta + 2; \gamma + 4; x) \quad F_4 - F_3 = \delta_4 x F_5
\]

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\[ G_0 = \frac{F_1}{F_0} \quad G_0 - 1 = \delta_1 x G_1 G_0 \quad G_0 = \frac{1}{1 - \delta_1 x G_1} \]
\[ G_1 = \frac{F_2}{F_1} \quad G_1 - 1 = \delta_2 x G_2 G_1 \quad G_1 = \frac{1}{1 - \delta_2 x G_2} \]
\[ G_2 = \frac{F_3}{F_2} \quad G_2 - 1 = \delta_3 x G_3 G_2 \quad G_2 = \frac{1}{1 - \delta_3 x G_3} \]

\[
\frac{F(\alpha, \beta + 1; \gamma + 1; x)}{F(\alpha, \beta; \gamma; x)} = G_0 = \frac{1}{1 - \delta_1 x G_1} = \frac{1}{\delta_1 x} \frac{1}{1 - \delta_2 x G_2} = \frac{1}{\delta_1 x} \frac{1}{1 - \delta_2 x} \frac{1}{1 - \delta_3 x G_3} 
\]

Since \( F(\alpha, 0; \gamma; x) = 1 \), we have developed a continued fraction expansion for \( F(\alpha, 1; \gamma + 1; x) \)

**Examples**

\[
\log(1 + x) = x F(1, 1; 2; -x) \\
\arctan x = x F(\frac{1}{2}, 1; \frac{3}{2}; -x^2) \\
\arcsin x = x F(\frac{1}{2}, \frac{3}{2}; x^2) = x \sqrt{1 - x^2} F(1, 1; \frac{3}{2}; x^2) \\
Q = \frac{2 \psi - \sin 2 \psi}{\sin^2 \psi} = \frac{4}{3} F(3, 1; \frac{5}{2}; \sin^2 \frac{1}{2} \psi) \\
\arctanh x = x F(\frac{1}{2}, 1; \frac{3}{2}; x^2)
\]

**Sufficient Conditions for Convergence of Continued Fractions**

**Class I**

\[
\frac{a_0}{b_0 + \frac{a_1}{b_1 + \frac{a_2}{b_2 + \frac{a_3}{b_3 + \ddots}}}}
\]

Will either converge or oscillate between two different values.

\[
\lim_{n \to \infty} \frac{b_n - 1 b_n}{a_n} > 0
\]

**Class II**

\[
\frac{a_0}{b_0 - \frac{a_1}{b_1 - \frac{a_2}{b_2 - \frac{a_3}{b_3 - \ddots}}}}
\]

Will either converge or diverge to infinity.

\[
b_n \geq a_n + 1
\]

**Note:** All \( a_n \) and \( b_n \) are positive.

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The Top-Down Method for Evaluating Continued Fractions

For a Class II continued fraction with \( n = 1, 2, \ldots \), we have

\[
\delta_n = \frac{1}{1 - \frac{a_n}{b_{n-1} b_n \delta_{n-1}}} \quad u_n = u_{n-1} (\delta_n - 1) \quad \Sigma_n = \Sigma_{n-1} + u_n
\]

where

\[
\delta_0 = 1 \quad u_0 = \Sigma_0 = \frac{a_0}{b_0}
\]

Continued Fractions Versus Power Series

For the tangent function

\[
\tan x = x + \frac{1}{3} x^3 + \frac{2}{15} x^5 + \frac{17}{315} x^7 + \frac{62}{2,835} x^9 + \cdots = \frac{x}{1 - \frac{x^2}{3 - \frac{x^2}{5 - \frac{x^2}{7 - \cdots}}}}
\]

\textbf{a.} The series converges for \(-\frac{1}{2} \pi \leq x \leq \frac{1}{2} \pi\).

\textbf{b.} The continued fraction converges for all \( x \) not equal to \( \frac{1}{2} \pi \pm n \pi \).