16.346 Astrodynamics
Fall 2008

## Numerical Integration

$$\frac{d\mathbf{r}}{dt} = \mathbf{v} \qquad \frac{d\mathbf{v}}{dt} = \mathbf{g}(\mathbf{r}) \quad \Longrightarrow \quad \frac{d\mathbf{x}}{dt} = \mathbf{y} \qquad \frac{d\mathbf{y}}{dt} = \mathbf{f}(\mathbf{x}) \quad \Longleftrightarrow \quad \frac{dx^i}{dt} = y^i \qquad \frac{dy^i}{dt} = f^i(x^i) \qquad i = 1,2,3$$

$$\mathbf{x}(t_0) = \mathbf{x}_0 \quad \mathbf{y}(t_0) = \mathbf{y}_0 \quad \mathbf{f}(\mathbf{x}_0) = \mathbf{f}_0$$

## Taylor Series Expansion

$$\mathbf{x}(t+h) = \mathbf{x}_0 + h\,\frac{d\mathbf{x}}{dt}\bigg|_{t=t_0} + \frac{h^2}{2!}\,\frac{d^2\mathbf{x}}{dt^2}\bigg|_{t=t_0} + \cdots = \mathbf{x}_0 + h\mathbf{y}_0 + \frac{h^2}{2!}\mathbf{f}_0 + \cdots$$

$$\mathbf{y}(t+h) = \mathbf{y}_0 + h\,\frac{d\mathbf{y}}{dt}\bigg|_{t=t_0} + \frac{h^2}{2!}\,\frac{d^2\mathbf{y}}{dt^2}\bigg|_{t=t_0} + \cdots = \mathbf{y}_0 + h\mathbf{f}_0 + \frac{h^2}{2!}\mathbf{f}_0' + \cdots$$

$$= \mathbf{y}_0 + h\mathbf{f}_0 + \frac{h^2}{2!}\underbrace{\frac{\partial \mathbf{f}}{\partial \mathbf{x}}\frac{d\mathbf{x}}{dt}\bigg|_{t=t_0}}_{=\mathbf{F}_0\mathbf{y}_0} + \cdots$$

**First Order Method**

$$\mathbf{x} = \mathbf{x}_0 + h\mathbf{y}_0 + O(h^2)$$
$$\mathbf{y} = \mathbf{y}_0 + h\mathbf{f}_0 + O(h^2)$$

**Second Order Method**

$$\mathbf{x} = \mathbf{x}_0 + h\mathbf{y}_0 + \tfrac{1}{2}h^2\mathbf{f}_0 + O(h^3)$$
$$\mathbf{y} = \mathbf{y}_0 + h\mathbf{f}_0 + \tfrac{1}{2}h^2\mathbf{F}_0\mathbf{y}_0 + O(h^3)$$

## How to Avoid Calculating the Matrix $\mathbf{F}_0$

$$\mathbf{f}(\mathbf{x}_0 + \delta\mathbf{x}) = \mathbf{f}_0 + \frac{\partial \mathbf{f}}{\partial \mathbf{x}}\bigg|_{t=t_0}\delta\mathbf{x} + O[(\delta x)^2]$$

or for any constant $p$

$$\mathbf{f}(\mathbf{x}_0 + hp\mathbf{y}_0) = \mathbf{f}_0 + \frac{\partial \mathbf{f}}{\partial \mathbf{x}}\bigg|_{t=t_0} hp\mathbf{y}_0 = \mathbf{f}_0 + hp\mathbf{F}_0\mathbf{y}_0 + O(h^2)$$

or

$$\frac{1}{p}[\mathbf{f}(\mathbf{x}_0 + hp\mathbf{y}_0) - \mathbf{f}_0] = h\mathbf{F}_0\mathbf{y}_0 + O(h^2)$$

Hence, the second-order method is equivalent to

$$\mathbf{x} = \mathbf{x}_0 + h\mathbf{y}_0 + \tfrac{1}{2}h^2\mathbf{f}(\mathbf{x}_0) + O(h^3)$$
$$\mathbf{y} = \mathbf{y}_0 + h\Big(1 - \frac{1}{2p}\Big)\mathbf{f}(\mathbf{x}_0) + \frac{1}{2p}h\,\mathbf{f}(\mathbf{x}_0 + hp\mathbf{y}_0) + O(h^3)$$

Choose $p = \frac{1}{2}$ and note that $\mathbf{f}(\mathbf{x}_0) - \mathbf{f}(\mathbf{x}_0 + hp\mathbf{y}_0) = O(h)$. Therefore, we have

$$\mathbf{x} = \mathbf{x}_0 + h\mathbf{y}_0 + \tfrac{1}{2}h^2\mathbf{f}(\mathbf{x}_0 + \tfrac{1}{2}h\mathbf{y}_0) + O(h^3)$$
$$\mathbf{y} = \mathbf{y}_0 + h\mathbf{f}(\mathbf{x}_0 + \tfrac{1}{2}h\mathbf{y}_0) + O(h^3)$$

with only one evaluation of the function $\mathbf{f}(\mathbf{x})$ for $\mathbf{x} = \mathbf{x}_0 + \frac{1}{2}h\mathbf{y}_0$.

## Formal Derivation of the Second-Order Method

Choose $a$, $b$, $p$ so that

$$\begin{aligned} \mathbf{x} &= \mathbf{x}_0 + h\mathbf{y}_0 + h^2 a\mathbf{k} + O(h^3) \\ \mathbf{y} &= \mathbf{y}_0 + hb\mathbf{k} + O(h^3) \\ \mathbf{k} &= \mathbf{f}(\mathbf{x}_0 + hp\mathbf{y}_0) \end{aligned} \qquad \equiv \qquad \begin{aligned} \mathbf{x} &= \mathbf{x}_0 + h\mathbf{y}_0 + \tfrac{1}{2}h^2\boldsymbol{\alpha}_0 + O(h^3) \\ \mathbf{y} &= \mathbf{y}_0 + h(\boldsymbol{\alpha}_0 + \tfrac{1}{2}h\boldsymbol{\alpha}_1) + O(h^3) \\ \mathbf{k} &= \boldsymbol{\alpha}_0 + hp\boldsymbol{\alpha}_1 + O(h^2) \end{aligned}$$

where $\boldsymbol{\alpha}_0 = \mathbf{f}_0$ and $\boldsymbol{\alpha}_1 = \mathbf{f}'_0 = \mathbf{F}_0\mathbf{y}_0$.
Expand $\mathbf{f}(\mathbf{x}_0 + hp\mathbf{y}_0)$ in a Taylor series:

$$\mathbf{f}(\mathbf{x}+\delta\mathbf{x}) = \mathbf{f}_0 + \left.\frac{\partial \mathbf{f}}{\partial \mathbf{x}}\right|_{t=t_0} \delta\mathbf{x} + O[(\delta x)^2] \quad \Longrightarrow \quad \mathbf{f}(\mathbf{x}_0 + hp\mathbf{y}_0) = \underbrace{\mathbf{f}_0 + \left.\frac{\partial \mathbf{f}}{\partial \mathbf{x}}\right|_{t=t_0} hp\mathbf{y}_0}_{=\,\boldsymbol{\alpha}_0 + hp\boldsymbol{\alpha}_1} + O(h^2)$$

Then

$$\begin{aligned} \mathbf{x} &= \mathbf{x}_0 + h\mathbf{y}_0 + h^2 a(\boldsymbol{\alpha}_0 + hp\boldsymbol{\alpha}_1) + O(h^3) \\ \mathbf{y} &= \mathbf{y}_0 + hb(\boldsymbol{\alpha}_0 + hp\boldsymbol{\alpha}_1) + O(h^3) \end{aligned} \qquad \equiv \qquad \begin{aligned} \mathbf{x} &= \mathbf{x}_0 + h\mathbf{y}_0 + \tfrac{1}{2}h^2\boldsymbol{\alpha}_0 + O(h^3) \\ \mathbf{y} &= \mathbf{y}_0 + h(\boldsymbol{\alpha}_0 + \tfrac{1}{2}h\boldsymbol{\alpha}_1) + O(h^3) \end{aligned}$$

Equate corresponding coefficients of $\boldsymbol{\alpha}_0$ and $\boldsymbol{\alpha}_1$

$$a = \tfrac{1}{2} \qquad \begin{bmatrix} 1 \\ p \end{bmatrix} b = \begin{bmatrix} 1 \\ \tfrac{1}{2} \end{bmatrix} \qquad \Longrightarrow \qquad a = p = \tfrac{1}{2} \qquad b = 1$$

In summary:

$$\boxed{\begin{aligned} \mathbf{x} &= \mathbf{x}_0 + h\mathbf{y}_0 + \tfrac{1}{2}h^2\mathbf{k} + O(h^3) \\ \mathbf{y} &= \mathbf{y}_0 + h\mathbf{k} + O(h^3) \\ \mathbf{k} &= \mathbf{f}(\mathbf{x}_0 + \tfrac{1}{2}h\mathbf{y}_0) \end{aligned}}$$

## Taylor Expansion using Indicial Notation and Summation Convention

$$\begin{aligned} x^i(t+h) &= x^i + h\frac{dx^i}{dt} + \frac{h^2}{2!}\frac{d^2x^i}{dt^2} + \frac{h^3}{3!}\frac{d^3x^i}{dt^3} + \frac{h^4}{4!}\frac{d^4x^i}{dt^4} + \cdots \\ &= x^i + hy^i + \frac{h^2}{2!}f^i + \frac{h^3}{3!}\frac{df^i}{dt} + \frac{h^4}{4!}\frac{d^2f^i}{dt^2} + \cdots \\ &= x^i + hy^i + \frac{h^2}{2!}f^i + \frac{h^3}{3!}\sum_{j=1}^{3}\frac{\partial f^i}{\partial x^j}\frac{dx^j}{dt} + \frac{h^4}{4!}\sum_{j=1}^{3}\frac{d}{dt}\left(\frac{\partial f^i}{\partial x^j}\frac{dx^j}{dt}\right) + \cdots \\ &= x^i + hy^i + \frac{h^2}{2!}f^i + \frac{h^3}{3!}f^i_j y^j + \frac{h^4}{4!}(f^i_{jk}y^j y^k + f^i_j f^j) + \cdots \end{aligned}$$

## Taylor Expansion of a Vector Function of a Vector

$$\boxed{f^i(x^i + \delta^i) = f^i + f^i_j\delta^j + \tfrac{1}{2}f^i_{jk}\delta^j\delta^k + \tfrac{1}{6}f^i_{jk\ell}\delta^j\delta^k\delta^\ell + \cdots}$$

$$\begin{aligned}
\mathbf{x} &= \mathbf{x}_0 + h\mathbf{y}_0 + h^2(a_0\mathbf{k}_0 + a_1\mathbf{k}_1) + O(h^4) \\
\mathbf{y} &= \mathbf{y}_0 + h\,(b_0\mathbf{k}_0 + b_1\mathbf{k}_1) + O(h^4) \\
\mathbf{k}_0 &= \mathbf{f}(\mathbf{x}_0 + hp_0\mathbf{y}_0) \\
\mathbf{k}_1 &= \mathbf{f}(\mathbf{x}_0 + hp_1\mathbf{y}_0 + h^2q_1\mathbf{k}_0)
\end{aligned}$$

## Series Expansion Using Indicial Notation

$$\begin{aligned}
x^i(t+h) &= x^i + hy^i + \tfrac{1}{2}h^2f^i + \tfrac{1}{6}h^3f^i_jy^j + O(h^4) \\
y^i(t+h) &= y^i + hf^i + \tfrac{1}{2}h^2f^i_jy^j + \tfrac{1}{6}h^3(f^i_{jk}y^jy^k + f^i_jf^j) + O(h^4)
\end{aligned}$$

$$k^i_0 = f^i(x^i + \underbrace{hp_0y^i}_{\delta^i_0}) = f^i + f^i_jhp_0y^j + \tfrac{1}{2}f^i_{jk}(hp_0y^j)(hp_0y^k) + O(h^3)$$

$$\begin{aligned}
k^i_1 = f^i(x^i + \underbrace{hp_1y^i + h^2q_1k^i_0}_{\delta^i_1}) &= f^i[x^i + hp_1y^i + h^2q_1f^i + O(h^3)] \\
&= f^i + f^i_j(hp_1y^j + h^2q_1f^j) + \tfrac{1}{2}f^i_{jk}(hp_1y^j)(hp_1y^k) + O(h^3)
\end{aligned}$$

## Series Expansion Using Vector Notation

$$\begin{aligned}
\mathbf{x} &= \mathbf{x}_0 + h\mathbf{y}_0 + h^2(\tfrac{1}{2}\boldsymbol{\alpha}_0 + \tfrac{1}{6}h\boldsymbol{\alpha}_1) + O(h^4) \\
\mathbf{y} &= \mathbf{y}_0 + h[\boldsymbol{\alpha}_0 + \tfrac{1}{2}h\boldsymbol{\alpha}_1 + \tfrac{1}{6}h^2(\boldsymbol{\alpha}_2 + \boldsymbol{\beta}_2)] + O(h^4) \\
\mathbf{k}_0 = \mathbf{f}(\mathbf{x}_0 + hp_0\mathbf{y}_0) &= \boldsymbol{\alpha}_0 + hp_0\boldsymbol{\alpha}_1 + \tfrac{1}{2}h^2p_0^2\boldsymbol{\alpha}_2 + O(h^3) \\
\mathbf{k}_1 = \mathbf{f}(\mathbf{x}_0 + hp_1\mathbf{y}_0 + h^2q_1\mathbf{k}_0) &= \boldsymbol{\alpha}_0 + hp_1\boldsymbol{\alpha}_1 + h^2(\tfrac{1}{2}p_1^2\boldsymbol{\alpha}_2 + q_1\boldsymbol{\beta}_2) + O(h^3)
\end{aligned}$$

## Determine $a_0,\ a_1,\ b_0,\ b_1,\ p_0,\ p_1,\ q_1$

$$\begin{aligned}
h^2(a_0\mathbf{k}_0 + a_1\mathbf{k}_1) &\equiv h^2(\tfrac{1}{2}\boldsymbol{\alpha}_0 + \tfrac{1}{6}h\boldsymbol{\alpha}_1) \quad \text{to terms of order } h^3 \\
h(b_0\mathbf{k}_0 + b_1\mathbf{k}_1) &\equiv h[\boldsymbol{\alpha}_0 + \tfrac{1}{2}h\boldsymbol{\alpha}_1 + \tfrac{1}{6}h^2(\boldsymbol{\alpha}_2 + \boldsymbol{\beta}_2)] \quad \text{to terms of order } h^3
\end{aligned}$$

## Condition Equations

$(\boldsymbol{\alpha})$

$$\begin{bmatrix} 1 & 1 \\ p_0 & p_1 \end{bmatrix}\begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{6} \end{bmatrix} \qquad \begin{bmatrix} 1 & 1 \\ p_0 & p_1 \\ p_0^2 & p_1^2 \end{bmatrix}\begin{bmatrix} b_0 \\ b_1 \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{1}{2} \\ \frac{1}{3} \end{bmatrix}$$

$(\boldsymbol{\beta})$

$$q_1b_1 = \tfrac{1}{2}(\tfrac{1}{3})$$

## Solving the Condition Equations

First

$$\begin{aligned} b_0 + b_1 &= 1 \\ p_0 b_0 + p_1 b_1 &= \tfrac{1}{2} \\ p_0^2 b_0 + p_1^2 b_1 &= \tfrac{1}{3} \end{aligned} \quad \Longrightarrow \quad \begin{aligned} (1-p_0)b_0 + (1-p_1)b_1 &= \tfrac{1}{2} \\ p_0(1-p_0)b_0 + p_1(1-p_1)b_1 &= \tfrac{1}{6} \end{aligned} \quad \Longleftrightarrow \quad \begin{aligned} a_0 + a_1 &= \tfrac{1}{2} \\ p_0 a_0 + p_1 a_1 &= \tfrac{1}{6} \end{aligned}$$

Hence $\quad a_0 = (1-p_0)b_0 \quad$ and $\quad a_1 = (1-p_1)b_1 \, .$

Next, for consistency,

$$\begin{aligned} b_0 + b_1 &= 1 \\ p_0 b_0 + p_1 b_1 &= \tfrac{1}{2} \\ p_0^2 b_0 + p_1^2 b_1 &= \tfrac{1}{3} \end{aligned} \quad \Longleftrightarrow \quad D_3 = \begin{vmatrix} 1 & 1 & 1 \\ p_0 & p_1 & \tfrac{1}{2} \\ p_0^2 & p_1^2 & \tfrac{1}{3} \end{vmatrix} = 0$$

Expand the determinant

$$D_3 = \begin{vmatrix} 1 & 1 & 1 \\ p_0 & p_1 & \tfrac{1}{2} \\ p_0^2 & p_1^2 & \tfrac{1}{3} \end{vmatrix} = \overbrace{(p_1 - p_0)}^{\text{Vandermonde determinant}} \underbrace{\left[\tfrac{1}{3} - \tfrac{1}{2}(p_0 + p_1) + p_0 p_1\right]}_{\text{Constraint function}} = V_2 \underbrace{L_3(p_0, p_1)}_{=0}$$

Complete solution

$$b_0 = \frac{\frac{1}{2} - p_1}{p_0 - p_1} \qquad b_1 = \frac{\frac{1}{2} - p_0}{p_1 - p_0} \qquad q_1 = \frac{1}{6}\,\frac{p_1 - p_0}{\frac{1}{2} - p_0} \qquad a_0 = (1-p_0)b_0 \qquad a_1 = (1-p_1)b_1$$

with $p_0$ and $p_1$ chosen arbitrarily subject to

$$L_3(p_0, p_1) = \tfrac{1}{3} - \tfrac{1}{2}(p_0 + p_1) + p_0 p_1 = 0$$

## Nyström's Third Order Algorithm with $p_0 = 0$

$$\boxed{\begin{aligned} \mathbf{x} &= \mathbf{x}_0 + h\mathbf{y}_0 + \tfrac{1}{4}h^2(\mathbf{k}_0 + \mathbf{k}_1) + O(h^4) \\ \mathbf{y} &= \mathbf{y}_0 + \tfrac{1}{4}h(\mathbf{k}_0 + 3\mathbf{k}_1) + O(h^4) \\ \mathbf{k}_0 &= \mathbf{f}(\mathbf{x}_0) \\ \mathbf{k}_1 &= \mathbf{f}(\mathbf{x}_0 + \tfrac{2}{3}h\mathbf{y}_0 + \tfrac{2}{9}h^2\mathbf{k}_0) \end{aligned}}$$

**Note:** In the book *Discrete Variable Methods in Ordinary Differential Equations* by Peter Henrici published by John Wiley & Sons, Inc. in 1962, there is a mistake. In his book, Henrici didn't derive the algorithm. He simply stated it. The factor $\frac{2}{9}$ was erroneously written as $\frac{1}{3}$. This error would be difficult to find since the algorithm would not exhibit any particular problem. It just would not be as accurate as it should be.

The moral is **Never copy somebody's algorithm!!** *Always derive it for yourself.*

## Nyström's Fourth Order Algorithm

### Used in the Apollo Guidance Computer

$$\mathbf{x} = \mathbf{x}_0 + h\mathbf{y}_0 + \tfrac{1}{6}h^2(\mathbf{k}_0 + 2\mathbf{k}_1) + O(h^5)$$
$$\mathbf{y} = \mathbf{y}_0 + \tfrac{1}{6}h(\mathbf{k}_0 + 4\mathbf{k}_1 + \mathbf{k}_2) + O(h^5)$$

where

$$\mathbf{k}_0 = \mathbf{f}(t_0, \mathbf{x}_0)$$
$$\mathbf{k}_1 = \mathbf{f}(t_0 + \tfrac{1}{2}h, \mathbf{x}_0 + \tfrac{1}{2}h\mathbf{y}_0 + \tfrac{1}{8}h^2\mathbf{k}_0)$$
$$\mathbf{k}_2 = \mathbf{f}(t_0 + h, \mathbf{x}_0 + h\mathbf{y}_0 + \tfrac{1}{2}h^2\mathbf{k}_1)$$

## Nyström's Fifth Order Algorithm

$$\mathbf{x} = \mathbf{x}_0 + h\mathbf{y}_0 + \tfrac{1}{192}h^2(23\mathbf{k}_0 + 75\mathbf{k}_1 - 27\mathbf{k}_2 + 25\mathbf{k}_3) + O(h^6)$$
$$\mathbf{y} = \mathbf{y}_0 + \tfrac{1}{192}h(23\mathbf{k}_0 + 125\mathbf{k}_1 - 81\mathbf{k}_2 + 125\mathbf{k}_3) + O(h^6)$$

where

$$\mathbf{k}_0 = \mathbf{f}(t_0, \mathbf{x}_0)$$
$$\mathbf{k}_1 = \mathbf{f}(t_0 + \tfrac{2}{5}h, \mathbf{x}_0 + \tfrac{2}{5}h\mathbf{y}_0 + \tfrac{2}{25}h^2\mathbf{k}_0)$$
$$\mathbf{k}_2 = \mathbf{f}(t_0 + \tfrac{2}{3}h, \mathbf{x}_0 + \tfrac{2}{3}h\mathbf{y}_0 + \tfrac{2}{9}h^2\mathbf{k}_0)$$
$$\mathbf{k}_3 = \mathbf{f}[t_0 + \tfrac{4}{5}h, \mathbf{x}_0 + \tfrac{4}{5}h\mathbf{y}_0 + \tfrac{4}{25}h^2(\mathbf{k}_0 + \mathbf{k}_1)]$$

## R-K-N Sixth Order Algorithm

$$\mathbf{x} = \mathbf{x}_0 + h\mathbf{y}_0 + \tfrac{1}{90}h^2(7\mathbf{k}_0 + 24\mathbf{k}_1 + 6\mathbf{k}_2 + 8\mathbf{k}_3) + O(h^7)$$
$$\mathbf{y} = \mathbf{y}_0 + \tfrac{1}{90}h(7\mathbf{k}_0 + 32\mathbf{k}_1 + 12\mathbf{k}_2 + 32\mathbf{k}_3 + 7\mathbf{k}_4) + O(h^7)$$

where

$$\mathbf{k}_0 = \mathbf{f}(t_0, \mathbf{x}_0)$$
$$\mathbf{k}_1 = \mathbf{f}(t_0 + \tfrac{1}{4}h, \mathbf{x}_0 + \tfrac{1}{4}h\mathbf{y}_0 + \tfrac{1}{32}h^2\mathbf{k}_0)$$
$$\mathbf{k}_2 = \mathbf{f}[t_0 + \tfrac{1}{2}h, \mathbf{x}_0 + \tfrac{1}{2}h\mathbf{y}_0 - \tfrac{1}{24}h^2(\mathbf{k}_0 - 4\mathbf{k}_1)]$$
$$\mathbf{k}_3 = \mathbf{f}[t_0 + \tfrac{3}{4}h, \mathbf{x}_0 + \tfrac{3}{4}h\mathbf{y}_0 + \tfrac{1}{32}h^2(3\mathbf{k}_0 + 4\mathbf{k}_1 + 2\mathbf{k}_2)]$$
$$\mathbf{k}_4 = \mathbf{f}[t_0 + h, \mathbf{x}_0 + h\mathbf{y}_0 + \tfrac{1}{14}h^2(6\mathbf{k}_1 - \mathbf{k}_2 + 2\mathbf{k}_3)]$$