Lectures 7: Modulation with 2-D signal

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Two-dimensional signals

- \( S_i = (S_{i1}, S_{i2}) \)

- Set of signal points is called a constellation

- 2-D constellations are commonly used

- Large constellations can be used to transmit many bits per symbol
  - More bandwidth efficient
  - More error prone

- The “shape” of the constellation can be used to minimize error probability by keeping symbols as far apart as possible

- Common constellations
  - QAM: Quadrature Amplitude Modulation
    - PAM in two dimensions
  - PSK: Phase Shift Keying
    - Special constellation where all symbols have equal power
Symmetric M-QAM

\[ S_m = (A_m^x, A_m^y), \quad A_m^x, A_m^y \in \left\{ \pm 1, \pm 3, \ldots, \pm (\sqrt{M} - 1) \right\} \]

M is the total number of signal points (symbols)

\( \sqrt{M} \) signal levels on each axis

Constellation is symmetric

\[ \Rightarrow M = K^2, \text{ for some } K \]

Signal levels on each axis are

the same as for PAM

E.g., 4-QAM \( \Rightarrow A_m^x, A_m^y \in \{\pm 1\} \)

16-QAM \( \Rightarrow A_m^x, A_m^y \in \{\pm 1, \pm 3\} \)
Bandwidth occupancy of QAM

- When using a rectangular pulse, the Fourier transform is a Sinc

\[ g(t) \]

\[ |G(f)| \]

- First null BW is still \( \frac{2}{T} \)
  - \( \log_2(M) \) bits per symbol
  - \( R_b = \frac{\log_2(M)}{T} \)
  - Bandwidth Efficiency = \( \frac{R_b}{BW} = \frac{\log_2(M)}{2} \)

\[ \Rightarrow \text{“Same as for PAM”} \]

But as we will see next, QAM is more energy efficient than PAM
Energy efficiency

\[ E_{sm} = [(A_m^x)^2 + (A_m^y)^2]E_g \]

\[ E[(A_m^x)^2] = E[(A_m^y)^2] = \frac{K^2 - 1}{3} = \frac{M - 1}{3}, \quad K = \sqrt{M} \]

\[ \overline{E_s} = \frac{2(M - 1)}{3} E_g \]

Transmitted energy \[= \frac{\overline{E_s}}{2} = \frac{(M - 1)}{3} E_g \]

\[ E_b(QAM) = \text{Energy / bit} = \frac{(M - 1)}{3 \log_2(M)} E_g \]

- Compare to PAM: \( E_b \) increases with \( M \), but not nearly as fast as PAM

\[ E_b(PAM) = \frac{(M^2 - 1)}{6 \log_2(M)} E_g \]
Bandpass QAM

- **Modulate the two dimensional signal by multiplication by orthogonal carriers (sinusoids): Sine and Cosine**
  - This is accomplished by multiplying the $A^x$ component by Cosine and the $A^y$ component by sine
  - Typically, people do not refer to these components as $x,y$ but rather $A^c$ or $A^s$ for cosine and sine or sometimes as $A^Q$, and $A^I$ for quadrature or in-phase components

- **The transmitted signal, corresponding to the $m^{th}$ symbol is:**

  \[ U_m(t) = A^x_m g(t) \cos(2\pi f_c t) + A^y_m g(t) \sin(2\pi f_c t), \quad m = 1 \ldots M \]
Modulator

Binary data → Map Log(M) bits into one of M symbols $A_m = (A^x, A^y)$ → g(t) → Cos($2\pi f_c t$) → $U_m(t)$

$U_m(t) = g(t) \cdot \text{Map Log}(M)$

$g(t) = \begin{cases} 
\text{Cos}(2\pi f_c t) & \text{if} \, \text{Binary data is} \, \text{Cos}
\end{cases}$

$g(t) = \begin{cases} 
\text{Sin}(2\pi f_c t) & \text{if} \, \text{Binary data is} \, \text{Sin}
\end{cases}$
Demodulation: Recovering the baseband signals

- Over a symbol duration, \( \sin(2\pi f_c t) \) and \( \cos(2\pi f_c t) \) are orthogonal
  - As long as the symbol duration is an integer number of cycles of the carrier wave \( f_c = n/T \) for some \( n \)
- When multiplied by a sine, the cosine component of \( U(t) \) disappears and similarly the sine component disappears when multiplied by cosine
Demodulation, cont.

\[ U(t)2\cos(2\pi f_c t) = 2A^x g(t)\cos^2(2\pi f_c t) + 2A^y g(t)\cos(2\pi f_c t)\sin\left(2\pi f_c t\right) \]

\[ \cos^2(\alpha) = \frac{1 + \cos(2\alpha)}{2} \]

\[ \Rightarrow U(t)2\cos(2\pi f_c t) = S^x(t) + S^x(t)\cos(4\pi f_c t) \rightarrow LPF \Rightarrow S^x(t) = A^x g(t) \]

Similarly,

\[ U(t)2\sin(2\pi f_c t) = 2A^x g(t)\cos(2\pi f_c t)\sin(2\pi f_c t) + 2A^y g(t)\sin^2(2\pi f_c t) \]

\[ \sin^2(\alpha) = \frac{1 - \cos(2\alpha)}{2} \]

\[ \Rightarrow U(t)2\sin(2\pi f_c t) = S^y(t) - S^y(t)\cos(4\pi f_c t) \rightarrow LPF \Rightarrow S^y(t) = A^y g(t) \]
Phase Shift Keying (PSK)

- Two Dimensional signals where all symbols have equal energy levels
  - I.e., they lie on a circle or radius $\sqrt{E_s}$

- Symbols are equally spaced to minimize likelihood of errors

- E.g., Binary PSK

- 4-PSK (above) same as 4-QAM
M-PSK

\[ A_i^x = \cos\left(\frac{2\pi i}{M}\right), \quad A_i^y = \sin\left(\frac{2\pi i}{M}\right), \quad i = 0, \ldots, M - 1 \]

\[ U_m(t) = g(t) A_m^x \cos(2\pi f_c t) - g(t) A_m^y \sin(2\pi f_c t) \]

Notice: \[ \cos(\alpha)\cos(\beta) = \frac{\cos(\alpha - \beta) + \cos(\alpha + \beta)}{2} \]

\[ \sin(\alpha)\sin(\beta) = \frac{\cos(\alpha - \beta) - \cos(\alpha + \beta)}{2} \]

Hence, \[ U_m(t) = g(t) \cos(2\pi f_c t + 2\pi n / M) \]

\[ \phi_m = 2\pi n / M = \text{phases shift of } m^{th} \text{ symbol} \]

\[ U_m(t) = g(t) \cos(2\pi f_c t + \phi_m), \quad m = 0 \ldots M - 1 \]
M-PSK Summary

- Constellation of M Phase shifted symbols
  - All have equal energy levels
  - \( \log_2(M) \) bits per symbol

- Modulation:

  \[ g(t) \]
  \[ \text{Map } \log(M) \text{ bits into one of } M \text{ symbols} \]
  \[ A_m = (A_x, A_y) \]

- Notice that for PSK we subtract the sine component from the cosine component
  - For convenience of notation only. If we added, the phase shift would have been negative but the end result is the same

- Demodulation is the same as for QAM