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Lecture 16: From Optimization To Estimation Theory and Back
Today

• Optimization examples

• Estimation Basics

Part I: Estimation Machinery
(more than what we need)
Example 1a: Triangulation (Structure Reconstruction)

Compute 3D point from known poses

\[ \lambda_1 \tilde{x}_1 = K_1[U_{w}^c t_{w}^{c1}] \tilde{p}^w \]

\[ \lambda_2 \tilde{x}_2 = K_2[U_{w}^c t_{w}^{c2}] \tilde{p}^w \]

Linear triangulation: \[ \min_{\| \tilde{p}^w \| = 1} \| A\tilde{p}^w \|^2 \]
Example 1b: Triangulation (Structure Reconstruction)

Compute 3D point from known poses

\[ \lambda_1 \tilde{x}_1 = K_1 [R_c^1, t_c^1] \tilde{p}^w \]

\[ \lambda_2 \tilde{x}_2 = K_2 [R_c^2, t_c^2] \tilde{p}^w \]

\[ \min_{p^w} \| x_1 - \pi(R_c^1, t_c^1, p^w) \|^2 + \| x_2 - \pi(R_c^2, t_c^2, p^w) \|^2 \]
Example 2a: Motion Estimation

\[
E_{12} = \arg\min_{E_{12} \in S_E} \sum_{k=1}^{N} |\tilde{y}_{k,2}^T E_{12} \tilde{y}_{k,1}|^2
\]

\[
E_{23} = \arg\min_{E_{23} \in S_E} \sum_{k=1}^{N} |\tilde{y}_{k,3}^T E_{23} \tilde{y}_{k,2}|^2
\]
Example 2b: Motion Estimation

Generalizes to K cameras: Bundle adjustment
Example 2b: Motion and Structure Estimation

Scale?

\[
\min_{(R_{ci}^w, t_{ci}^w), i=1,2,3, p_k^w, k=1,\ldots,N} \sum_{k=1}^{N} \sum_{i=1}^{3} \| x_{k,i} - \pi(R_{ci}^w, t_{ci}^w, p_k^w) \|^2
\]

Generalizes to K cameras: Bundle adjustment
Structure from Motion

180 cameras, 88723 points
458642 projections
active camera: 4

[courtesy of F. Dellaert & Y-D. Yian]
Estimation Theory

Concerned with the estimation of unknown variables given (noisy) measurements and prior information

**Estimator**: a function of the measurements that approximates the unknown variables

Measurements that depend on some unknown variable $x$:

$z_1, \ldots, z_N$

Estimator for $x$:

$x^* = \mathcal{F}(z_1, \ldots, z_N)$

$x^* \approx x$
Maximum Likelihood Estimation (MLE)

Assume we are given $N$ measurements $z_1, \ldots, z_N$ (e.g., pixel measurements) that are function of a variable we want to estimate $x$ (e.g., camera poses, points). Assume that we are also given the conditional distributions:

$$P(z_j|x)$$

Than the *maximum likelihood* estimator (MLE) is defined as:

$$x_{\text{MLE}} = \arg \max_x P(z_1, \ldots, z_N|x)$$

where $P(z_1, \ldots, z_N|x)$ is also called the *likelihood* of the measurements given $x$. Equivalently:

$$x_{\text{MLE}} = \arg \min_x -\log P(z_1, \ldots, z_N|x)$$
Maximum Likelihood Estimation (MLE)

Assume we are given $N$ measurements $z_1, \ldots, z_N$ (e.g., pixel measurements) that are function of a variable we want to estimate $x$ (e.g., camera poses, points). Assume that we are also given the conditional distributions:

$$P(z_j|x)$$

Then the maximum likelihood estimator (MLE) is defined as:

$$x_{\text{MLE}} = \arg \max_x \mathbb{P}(z_1, \ldots, z_N|x)$$

where $\mathbb{P}(z_1, \ldots, z_N|x)$ is also called the likelihood of the measurements given $x$. Equivalently:

$$x_{\text{MLE}} = \arg \min_x -\log \mathbb{P}(z_1, \ldots, z_N|x)$$
Maximum a Posteriori Estimation (MAP)

Assume we are given $N$ measurements $z_1, \ldots, z_N$ (e.g., pixel measurements) that are function of a variable we want to estimate $x$ (e.g., camera poses, points). Maximum a Posteriori Estimation (MAP) is a generalization of MLE. Then the MAP estimator is:

$$x_{MAP} = \arg \max_x P(x | z_1, \ldots, z_N)$$

Using Bayes rule:

$$x_{MAP} = \arg \max_x \frac{P(x | z_1, \ldots, z_N) P(x)}{P(z_1, \ldots, z_N)} = \arg \max_x \frac{P(x | z_1, \ldots, z_N) P(x)}{P(z_1, \ldots, z_N)} = \arg \max_x P(z_1, \ldots, z_N | x) P(x) \quad \text{Measurement likelihood Priors}$$
Maximum a Posteriori Estimation (MAP)

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Using Bayes rule:

$$x_{MAP} = \arg \max_x P(x|z_1, \ldots, z_N) = \arg \max_x \frac{P(z_1, \ldots, z_N|x)P(x)}{P(z_1, \ldots, z_N)} = \arg \max_x P(z_1, \ldots, z_N|x)P(x)$$

Assuming independence between measurements:

$$x_{MAP} = \arg \min_x -\sum_{j=1}^{N} \log P(z_j|x) - \log P(x)$$
Optimization

Linear triangulation:

$$\min_{\tilde{p}^w} \| A \tilde{p}^w \|^2 \quad \text{subject to} \quad \| \tilde{p}^w \| = 1$$

Nonlinear triangulation:

$$\min_{p^w} \| x_1 - \pi(R_{c_1}^w, t_{c_1}^w, p^w) \|^2 + \| x_2 - \pi(R_{c_2}^w, t_{c_2}^w, p^w) \|^2$$