Homework 1: Preliminary Design of a Satellite Launcher

a) Velocity Calculations:

Point A: Start of ascent trajectory  
Point B: Apogee of ascent trajectory

Conservation of Angular Momentum:

\[ v_1 \cos \alpha \cdot R_E = v_a \cdot R_c \]

Rearranging becomes:

\[ v_a = v_1 \frac{R_E}{R_c} \cos \alpha \quad (1) \]

Conservation of Energy:

\[
\frac{v_1^2}{2} = \frac{\mu_E}{R_E} = \frac{v_a^2}{2} = \frac{\mu_E}{R_E} = \frac{v_1^2}{2} \left( \frac{R_E}{R_c} \cos \alpha \right)^2 - \frac{\mu_E}{R_E} \\
\frac{1}{2} v_1^2 \left( 1 - \frac{R_E^2}{R_c^2} \cos^2 \alpha \right) = \mu_E \frac{v_a^2}{R_c} - \frac{\mu_E}{R_c} \\
v_1 = \sqrt{\frac{2\mu_E}{R_E} \left( \frac{1 - R_E \cos \alpha}{R_c} \right)} \quad (2)
\]

Substituting Values:

Using equation (2) to find \( v_1 \):

\[
v_1 = \sqrt{\frac{2 \cdot 3.98 \times 10^{14}}{6.37 \times 10^5} \left( \frac{1 - 6377/687 \cos 20^\circ}{1 - 6377/687 \cos 20^\circ} \right)} \\
v_1 = 6145 \text{ m/s}
\]

Using equation (1) to find \( v_a \):

\[
v_a = 6145 \times \frac{6377}{687} \cos 20^\circ \\
v_a = 5354 \text{ m/s}
\]

Orbital Velocity:

\[
v_c = \sqrt{\frac{\mu_E}{R_c}} \quad (3) \\
v_c = \sqrt{\frac{3.98 \times 10^{14}}{6.87 \times 10^6}} \\
v_c = 7611 \text{ m/s}
\]
**b) Stage ∆V Calculations:**

We allocate the full “apogee kick” to the third stage:

\[ \Delta V_3 = v_c - v_a \]  
\[ \Delta V_3 = \frac{7611}{s} - \frac{5354}{s} \]  
\[ \Delta V_3 = 2257 \frac{m}{s} \]

The initial ascent velocity \( v_1 \) contains the velocity increments of the 1st and 2nd stages, plus the contribution from Earth rotation, minus the losses from gravity and drag:

\[ v_1 = \Delta V_1 + \Delta V_2 + \omega_E R_E \cos \alpha - \Delta V_G - \Delta V_D \] (5)

**By design:**

\[ \Delta V_1 = \Delta V_2 = \frac{1}{2} (v_1 - \omega_E R_E \cos \alpha + \Delta V_G + \Delta V_D ) \]

**To solve:**

\[ \omega_E R_E \cos \alpha = 7.268e (-5) \frac{rad}{s} \times 6.37e6 \text{ m} \times \cos 20\degree \]
\[ \omega_E R_E \cos \alpha = 435 \frac{m}{s} \]

**To estimate the gravity loss** (1st stage only) we assume:

\[ \sin \gamma = 1 - \frac{t}{t_{b1}} \] (6)

(sin \( \gamma \) is linear between \( \gamma = 1 \) at \( t = 0 \) and \( \gamma = 0 \) at \( t = t_{b1} \))

**Determining relations among parameters:**

\[ t_{b1} = \frac{M_{p1}}{m_1} = \frac{M_{p1}}{(P_1/c)} = \frac{M_{p1}}{M_{01} * 3g/c} \]
\[ \frac{M_{p1}}{M_{01}} = 1 - \frac{M_{f1}}{M_{01}} = 1 - e^{-\frac{\Delta V_1}{c}} \]

**We need to make a preliminary guess at ∆V₁.** Take for now \( \Delta V_1 = 3200 \frac{m}{s} \).

\[ \frac{M_{p1}}{M_{01}} = 1 - e^{-3200 / (3*270)} = 0.7016 \]
\[ t_{b1} = \frac{0.7016}{3*9.8} (9.8 * 270) = 63.15 \text{ s} \]

**Solving for ∆V₆:**

Making the substitution \( z = \frac{t}{t_{b1}} \)

\[ \Delta V_G = \int_0^{t_{b1}} g \sin \gamma \, dt = gt_{b1} \int_0^1 (1 - z) \, dz = \frac{1}{2} gt_{b1} \] (7)

**As a first approximation:**

\[ \Delta V_G = \frac{1}{2} \times 9.8 \times 63.15 \]
\[ \Delta V_G = 309 \frac{m}{s} \]
We can now calculate a better $\Delta V_1$:

$$\Delta V_1 = \frac{1}{2} (6145 - 435 + 309 + 150)$$

$$\Delta V_1 = 3085 \frac{m}{s}$$

Refine other quantities:

$$\frac{\mu_{p1}}{\mu_{01}} = 1 - e^{-\frac{3085}{2646}} = 0.6884$$

$$t_{b1} = \frac{0.6884 \times 270}{3} = 61.95 \text{ s}$$

$$\Delta V_o = \frac{1}{2} \times 9.8 \times 61.95 = 304 \frac{m}{s}$$

$$\Delta V_1 = 3082 \frac{m}{s}$$

$$\Delta V_2 = \Delta V_1 = 3082 \frac{m}{s}$$

These values are close enough to the first approximation, and we accept them as converged.

c) Calculation of Stage Masses:

For each stage:

$$\frac{\mu_{pay,i}}{\mu_{0,i}} = e^{-\frac{\Delta V_i}{c}} = \varepsilon \ (8)$$

We apply this first to the $3^{rd}$ stage, for which $m_{pay,3} = m_{pay} = 3 \text{ kg}$.

$$M_{03} = \frac{3}{e^{-\frac{2257}{2646}-0.1}} = \frac{3}{0.3361} = 9.20 \text{ kg}$$

The structural mass of the third stage is then:

$$M_{s3} = 0.1M_{03} = 0.92 \text{ kg}$$

The propellant mass is:

$$M_{p3} = 9.20 \left(1 - e^{-\frac{2257}{2646}}\right) = 5.28 \text{ kg}$$

As a check: $M_{s3} + M_{p3} + M_{pay3} = 0.92 + 5.28 + 3 = 9.20 \text{ kg} = M_{03}$ (as it should)

For the second stage:

$$M_{pay2} = M_{03} = 9.20 \text{ kg}$$

$$M_{02} = \frac{9.20}{e^{-\frac{3082}{2646}-0.1}} = \frac{9.20}{0.2120} = 43.39 \text{ kg}$$

$$M_{s2} = 0.1 \times 43.39 = 4.34 \text{ kg}$$

$$M_{p2} = 43.39 \left(1 - e^{-\frac{3082}{2646}}\right) = 29.85 \text{ kg}$$

Again, we check that: $M_{s2} + M_{p2} + M_{pay2} = 4.34 + 29.85 + 9.20 = 43.39 \text{ kg} = M_{02}$
For the first stage:

\[ M_{\text{pay}4} = M_02 = 43.49 \, \text{kg} \]

\[ M_{01} = \frac{43.39}{e^{\frac{3082}{2646} - 0.1}} = \frac{43.39}{0.2120} = 204.67 \, \text{kg} \]

\[ M_{s1} = 0.1 \times 204.67 = 20.47 \, \text{kg} \]

\[ M_{p1} = 204.67 \left(1 - e^{-\frac{3082}{2646}}\right) = 140.82 \, \text{kg} \]

We verify that \( M_{s1} + M_{p1} + M_{\text{pay}1} = 20.47 + 140.82 + 43.39 = 204.68 \, \text{kg} \)

d) Thrusts and Firing Times:

\[ F_1 = M_{01} \times 3g = 204.67 \times 3 \times 9.8 = 6,017 \, \text{N} \]

\[ F_2 = M_{02} \times 3g = 43.39 \times 3 \times 9.8 = 1,276 \, \text{N} \]

\[ F_3 = M_{03} \times 3g = 9.20 \times 3 \times 9.8 = 270 \, \text{N} \]

Flow rates are then:

\[ \dot{m}_1 = \frac{6017}{2646} = 2.274 \frac{\text{kg}}{\text{s}} \]

\[ \dot{m}_2 = \frac{1276}{2646} = 0.4822 \frac{\text{kg}}{\text{s}} \]

\[ \dot{m}_1 = \frac{270}{2646} = 0.1020 \frac{\text{kg}}{\text{s}} \]

Firing times are given by \( t_{bi} = \frac{m_{pi}}{\dot{m}_i} \)

\[ t_{b1} = \frac{140.82}{2.274} = 61.93 \, \text{s} \]

\[ t_{b1} = \frac{29.85}{0.4822} = 61.91 \, \text{s} \]

\[ t_{b1} = \frac{5.28}{0.1020} = 51.76 \, \text{s} \]