Homework 2: Chemical vs. Electrical Thrusters

a) Chemical:
We start outside the sphere of influence (SOI) of Earth, and end outside the SOI of Mars, so no “escape” or “capture” $\Delta V$s are involved. The whole motion is under the Sun’s influence alone.

To enter the transfer orbit:

$$\Delta V_1 = v_{\text{perihelion}} - v_{c,\text{Earth}} = \frac{\mu_S}{r_E} \left( \sqrt{\frac{2r_M}{r_E + r_M}} - 1 \right)$$  \hspace{1cm} (1)$$

Known and calculated values:

$\mu_S = 1.327E20 \text{ m}^3 \text{s}^{-2}$
$r_E = 1.496E11 \text{ m}$
$r_M = 1.5237 \times r_E = 2.279E11 \text{ m}$
$v_{c,\text{Earth}} = 29,780 \text{ m s}^{-1}$
$v_{c,\text{Mars}} = 24,130 \text{ m s}^{-1}$

Substituting values:

$$\Delta V_1 = 29,780 \left( \sqrt{\frac{2+1.5237}{2.5237}} - 1 \right) = 2,945 \text{ m s}^{-1}$$

To enter circular orbit near Mars:

$$\Delta V_2 = v_{c,\text{Mars}} - v_{\text{apohelion}} = \frac{\mu_S}{r_M} \left( 1 - \sqrt{\frac{2r_E}{r_E + r_M}} \right)$$  \hspace{1cm} (2)$$

$$\Delta V_2 = 24,130 \left( 1 - \frac{2}{\sqrt{2.5237}} \right) = 2,649 \text{ m s}^{-1}$$

Total $\Delta V$: 

\[ \Delta V = \Delta V_1 + \Delta V_2 = 5.549 \text{ m/s (Chemical)} \]

Transfer duration:
The transfer time is \( \frac{1}{3} \) the orbital time in the transfer ellipse.

Semiaxis: \( a = \frac{r_{\text{initial}} + r_{\text{final}}}{2} = 1.8877E11 \text{ m} \)
\[
\Delta t = \frac{1}{2} \cdot 2\pi \cdot \frac{a^{3/2}}{\sqrt{\mu_s}} \quad (3)
\]
\[\Delta t = 2.237E7 \text{ s} = \frac{2.237E7}{86400} = 259 \text{ days} \]
\[
\frac{M_{\text{pay}}}{M_0} = e^{-\Delta V/c} - \varepsilon \quad (4)
\]
\[M_{\text{pay}} = 20,000 \left( e^{-\frac{5.547}{4.500}} \right) - 0.05 = 4,770 \text{ kg} \]

b) Electrical:
The propulsive \( \Delta V \) is now:
\[\Delta V = v_{\text{c,E}} - v_{\text{c,M}} = 29,780 - 24,130 = 5,650 \text{ m/s (Electric Propulsion)} \]

This is only slightly more than the chemical \( \Delta V \); for transfers to larger radii, the difference is more noticeable.

For optimization of the low-thrust mission, define non-dimensional variables:

\[\mu = \frac{M_{\text{pay}}}{M_0} \quad (5)\]
\[v = \frac{\Delta V}{c} \quad (6)\]
\[\lambda = \frac{a_0 \alpha \Delta V}{2 \eta} \quad (7)\]
\[\varepsilon = \frac{M_{\text{str}}}{M_0} \quad (8)\]

\( \alpha = 10 \frac{\text{kg}}{\text{kW}} = 0.01 \frac{\text{kg}}{\text{W}} \) is the specific mass (per unit power) of the power and propulsion equipment.

\( a_0 \) is the initial acceleration.

Combining expressions:
\[\mu = e^{-v} - \frac{\lambda}{v} - \varepsilon \quad (9)\]

To find the best specific impulse \( c \), we have differentiate with respect to \( v \):
\[-e^{-v} + \frac{\lambda}{v^2} = 0\]
\[ \lambda = v^2 e^{-v} \quad (10) \]

**Substituting values:**

\[ \lambda = \frac{0.01 \times 5650}{2 + 0.7} = 40.39 a_0 \]

For each value of \( a_0 \) we then need to solve (by trial and error) the equation:

\[ 40.39 a_0 = v_{\text{opt}}^2 e^{-v_{\text{opt}}} \quad (11) \]

Once \( v_{\text{opt}} \) is known, we calculate:

\[ c_{\text{opt}} = \frac{\Delta V}{v_{\text{opt}}} = \frac{5650}{v_{\text{opt}}} \]

The implied transfer time follows from:

\[ \Delta t = \frac{\mu_{\text{prop}}}{m} = \frac{M_0}{\frac{c}{v_{\text{opt}}} (1 - e^{-\frac{\Delta V}{c}})} = \frac{c}{a_0} \left( 1 - e^{-\frac{\Delta V}{c}} \right) \approx \frac{\Delta V}{a_0} \text{ if } \frac{\Delta V}{c} \ll c \quad (12) \]

The power per unit initial mass:

\[ \frac{P}{M_0} = \frac{\Delta V}{2\eta} = \frac{a_0 c}{2\eta} \quad (13) \]

Finally, the payload mass is:

\[ \mu_{\text{pay}} = \mu_{\text{opt}} M_0 = M_0 \left( e^{-v_{\text{opt}}} - \frac{\lambda}{v_{\text{opt}}} - \varepsilon \right) \quad (14) \]

The results are tabulated below for a range of initial accelerations:

<table>
<thead>
<tr>
<th>( a_0 ) [m/s^2]</th>
<th>( \lambda )</th>
<th>( v_{\text{opt}} )</th>
<th>( c_{\text{opt}} [m/s] )</th>
<th>( \Delta t [\text{days}] )</th>
<th>( \frac{P}{M_0} [W/kg] )</th>
<th>( M_{\text{pay}} [kg] )</th>
<th>( P [\text{MW}] )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1E-4</td>
<td>4.039E-3</td>
<td>0.06567</td>
<td>86052</td>
<td>633.3</td>
<td>6.149</td>
<td>16498</td>
<td>0.123</td>
</tr>
<tr>
<td>2E-4</td>
<td>8.078E-3</td>
<td>0.09421</td>
<td>60004</td>
<td>322.2</td>
<td>8.572</td>
<td>15486</td>
<td>0.1714</td>
</tr>
<tr>
<td>4E-4</td>
<td>0.01616</td>
<td>0.1361</td>
<td>41536</td>
<td>152.9</td>
<td>11.867</td>
<td>14082</td>
<td>0.2373</td>
</tr>
<tr>
<td>6E-4</td>
<td>0.02423</td>
<td>0.1694</td>
<td>33371</td>
<td>100.3</td>
<td>14.302</td>
<td>13024</td>
<td>0.286</td>
</tr>
</tbody>
</table>

We see several important things here:

a) For any specific power \( \frac{P}{M_0} \geq 10 \text{ W/kg} \), the transfer is faster than chemical.

b) The payload delivered is 3-4 times greater than with chemical.

c) The specific impulse is in the range from 3,400s to 8,700s.
d) The required power is from 120-290 KW, possible with large solar arrays.