Homework 4.1: Solid Propellant Rocket

1a) Normal operation mass balance:
\[ a \rho_p A_p P_{c0}^n - \frac{A_t P_{c0}}{c^*} = 0 \]  \hspace{1cm} (1)

Transient operation after port opening:
\[ \frac{V_c}{RT_c} \frac{dP_c}{dt} = a \rho_p A_p P_{c0}^n - \frac{(A_t + A_p) P_c}{c^*} \]  \hspace{1cm} (2)

Define:
\[ t_{ch} = \frac{V_c c^*}{RT_c A_b} \]  \hspace{1cm} (3)
\[ \tau = \frac{t}{t_c} \]  \hspace{1cm} (4)
\[ \alpha = \frac{A_p}{A_t} \]  \hspace{1cm} (5)
\[ y = \frac{P_c}{P_{c0}} \]  \hspace{1cm} (6)

In terms of these variables, equation (2) becomes:
\[ \frac{dy}{d\tau} = y^n - (1 + \alpha) y \]  \hspace{1cm} (7)

Multiply equation (7) by \( y^{-n} \):
\[ y^{-n} \frac{dy}{d\tau} = 1 - (1 + \alpha) y^{1-n} \]  \hspace{1cm} (8)
\[ (1 - n) y^{-n} dy = d(y^{1-n}) \]

Define:
\[ y^{1-n} = u \]  \hspace{1cm} (9)
\[ \frac{1}{1-n} \frac{du}{d\tau} = 1 - (1 + \alpha) u \]  \hspace{1cm} (10)

This relationship is linear and can be easily solved. The particular solution is \( u = \frac{1}{1+\alpha} \), and the homogeneous solution is \( u = e^{-(1-n)(1+\alpha)\tau} \). The complete solution is then \( u = \frac{1}{1+\alpha} + Ae^{-(1-n)(1+\alpha)\tau} \), where \( A \) is arbitrary.

At \( t = 0 \) (\( \tau = 0 \)), we have \( P_c = P_{c0} \) (\( y = u = 1 \)), so
\[ 1 = \frac{1}{1+\alpha} + A \]
\[ A = \frac{1}{1+\alpha} \]

Therefore:
\[ u = \frac{1 + \alpha e^{-(1-n)(1+\alpha)\tau}}{1 + \alpha} \]

**Using** \( y = u^{1-n} \):

\[ y = \left( \frac{1 + \alpha e^{-(1-n)(1+\alpha)\tau}}{1 + \alpha} \right)^{\frac{1}{1-n}} \] \( (11) \)

**1b)** The combustion stops when \( P_c \leq 20 \text{ atm} \) \( \left( y_{\text{extinction}} = \frac{20}{70} = \frac{2}{7} \right) \). For \( t \to \infty \) \( (\tau \to \infty) \), we obtain from equation \( (11) \):

\[ y(\infty) = \frac{1}{(1+\alpha)^{1-n}} \] \( (12) \)

For this to be equal or less than \( \frac{2}{7} \), \( \alpha \) must be more than:

\[ \alpha_{\text{min}} = \left( \frac{1}{y(\infty)} \right)^{1-n} - 1 = \left( \frac{7}{2} \right)^{1-0.2} - 1 \] \( (13) \)

\[ \left( \frac{A_p}{A_t} \right)_{\text{min}} = 1.724 \] \( (14) \)

For values of \( \alpha > 1.724 \), the extinction limit \( y = \frac{2}{7} \) is reached in a finite time. Solving equation \( (11) \) for \( \tau \) gives:

\[ \tau_{\text{ext}} = \frac{1}{(1+\alpha)(1-n)} \ln \left( \frac{\alpha}{(1+\alpha)y_{\text{ext}}^{1-n-1}} \right) \] \( (15) \)

Since \( n = 0.2 \) and \( (1 + \alpha)y_{\text{ext}}^{1-n} = \frac{(1+\alpha)}{(1+\alpha_{\text{min}})}(1 + \alpha_{\text{min}})y_{\text{ext}}^{1-n} \):

\[ \tau_{\text{ext}} = \frac{1}{0.8(1+\alpha)} \ln \left( \frac{\alpha}{(2.724 - 1)} \right) \] \( (16) \)

\[ t_{\text{ext}} = \tau_{\text{ext}} \frac{V_c^*}{R T_c A_t} \] \( (17) \)

Assuming a molecular mass \( M = 20 \text{ g mole}^{-1} \) (not specified in problem statement),

\[ \frac{V_c^*}{R T_c A_t} = \frac{10 \times 1800}{0.02 \times 3400} = 1.274 \times 10^{-2} \text{ s} \] \( (18) \)

We can now calculate a few extinction times corresponding to choices of \( \frac{A_p}{A_t} \) above the minimum, shown in Table 1.

**Table 1: Extinction Times**

<table>
<thead>
<tr>
<th>( \alpha = \frac{A_p}{A_t} )</th>
<th>1.724</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \tau_{\text{ext}} )</td>
<td>( \infty )</td>
<td>1.243</td>
<td>0.5803</td>
<td>0.3915</td>
</tr>
<tr>
<td>( t_{\text{ext}} \text{ [s]} )</td>
<td>( \infty )</td>
<td>1.583e-2</td>
<td>7.42e-3</td>
<td>4.99e-3</td>
</tr>
</tbody>
</table>
Homework 4.2: Monopropellant Hydrazine Rocket

Rapid disproportionation: \[ N_2H_4 \rightarrow \frac{4}{3} NH_3 + \frac{1}{3} N_2 \] (1)

Slow \( NH_3 \) decomposition: \[ \frac{4}{3} NH_3 \rightarrow \frac{2}{3} N_2 + 2H_2 \] (2)

Since we assume 40\% Ammonia decomposition, form equation (1) + 0.4*equation (2):

\[ N_2H_4 \rightarrow \frac{4}{3} (1 - 0.4) NH_3 + \frac{1+0.8}{3} N_2 + 0.4 * 2H_2 \] (3)

\[ N_2H_4 \rightarrow 0.8NH_3 + 0.6N_2 + 0.8H_2 \] (4)

We can now write the enthalpy balance for the reaction. The enthalpy of the reactants (liquid Hydrazine at 298.2K) is +50.63 kJ/mol, so using the fits provided for gaseous \( NH_3, N_2, \) and \( H_2 \):

\[ 50.63 = 0.8(-70.40 + 51.166\theta + 4.11\theta^2) + 0.6(-11.84 + 32.42\theta + 0.76\theta^2) + \ldots \]
\[ +0.8(-8.23 + 27.16\theta + 1.34\theta^2) \]

\[ 4.976\theta^2 + 82.508\theta - 120.64 = 0 \]

\[ \theta = \frac{-82.508 \pm \sqrt{82.508^2 + 4 \times 4.976 \times 120.64}}{2 \times 4.976} = 1.352 = \frac{T}{1000} \] (5)

\[ T = 1352K \]

Mean molecular mass:

\[ \bar{M} = \frac{0.8 \times 17 + 0.6 \times 28 + 0.8 \times 2}{0.8 + 0.6 + 0.8} = 16.0 \frac{g}{mol} = 0.016 \frac{kg}{mol} \]

Mean specific heat:

\[ c_p = \frac{0.8(c_p)_{NH_3} + 0.6(c_p)_{N_2} + 0.8(c_p)_{H_2}}{(0.8 + 0.6 + 0.8)} \]

\[ (c_p)_{NH_3} = \frac{\partial h_{NH_3}}{\partial T} = \frac{1}{1000} \frac{\partial h_{NH_3}}{\partial \theta} = \frac{51.66}{1000} \frac{kJ}{mol \cdot K} = 51.66 \frac{J}{mol \cdot K} \]

\[ (c_p)_{N_2} = \frac{\partial h_{N_2}}{\partial T} = \frac{1}{1000} \frac{\partial h_{N_2}}{\partial \theta} = \frac{32.42}{1000} \frac{kJ}{mol \cdot K} = 32.42 \frac{J}{mol \cdot K} \]

\[ (c_p)_{H_2} = \frac{\partial h_{H_2}}{\partial T} = \frac{1}{1000} \frac{\partial h_{H_2}}{\partial \theta} = \frac{27.61}{1000} \frac{kJ}{mol \cdot K} = 27.61 \frac{J}{mol \cdot K} \]

\[ c_p = \frac{0.4 \times 51.66 + 0.3 \times 32.42 + 0.4 \times 27.61}{0.016} = 2.590 \frac{J}{kg \cdot K} \]

We could now calculate \( c_p = 2.590 - \frac{0.314}{0.016} = 2.070 \frac{J}{kg \cdot K} \) and so \( \frac{c_p}{c_p} = 1.2512 \)