**Homework 6: Off-Design Performance of Small Turboprop Engine**

**a)** At the end of climb, $z = 6000 \text{ m}$, we have $T_0 = 261K \ (a_0 = 323.8 \text{ m/s})$, and $P_0 = 4.86 \times 10^4 \text{ Pa}$. Also, $M_0 = 0.6 \ (\theta_0 = 1 + 0.2 \times 0.6^2 = 1.072, T_{t0} = \theta_0 T_0 = 279.8K)$. Finally,

$$P_{t0} = P_0 \theta_0^{\gamma-1} = 6.199 \times 10^4 \text{ Pa}.$$ 

**Force balance along and transverse to the trajectory:**

$$L = W \cos \gamma \quad (1)$$

$$F = D + W \sin \gamma \quad (2)$$

From equation (2), $F = \frac{L}{L_D} + W \sin \gamma = W \left(\frac{\cos \gamma}{L_D} + \sin \gamma\right)$

We are given $W = 2000 \times 9.8 = 19600 \text{ N}, \gamma = 20^\circ, L_D = 15$, so

$$F = 19.600 \left(\frac{\cos 20^\circ}{15} + \sin 20^\circ\right) = 7931 \text{ N}$$

**b)** Since $T_{t4} = 1200 \text{ K}$, we have $\theta_t = \frac{1200}{261} = 4.598$.

The compressor ratio is chosen for maximum thrust, so

$$\tau_c = \sqrt[\gamma]{\theta_t} = \sqrt[\gamma]{9.598} = 2.0003 \quad (3)$$

$$T_{t3} = T_{t0} \tau_c = 559.7 \text{ K} \quad (4)$$

$$\pi_c = \tau_c^{\gamma-1} = (2.0003)^{3.5} = 11.32 \quad (5)$$

$$P_{t3} = P_{t0} \pi_c = 7.017 \times 10^5 \text{ Pa} \quad (6)$$

**From the shaft power balance:**

$$\tau_t = 1 - \frac{\tau_c^{\gamma-1}}{\theta_0} = 1 - \frac{1.0003}{4.598} (1.072) = 0.7668$$

$$\pi_t = \tau_t^{\gamma-1} = 0.3948$$

**From these results:**
c) To calculate the air flow rate \( \dot{m} \) we need the value of \( \frac{F}{ma_0} \). We assume here matched exhaust conditions and use:

\[
\frac{F}{ma_0} = \sqrt{\frac{2}{\gamma-1} \left( \theta_0 \tau_c - 1 \right) \frac{\theta_t}{\theta_0 \tau_c} - M_0} \quad (7)
\]

\[
\frac{F}{ma_0} = \sqrt{5(1.072 \times 2000 \times 0.7668 - 1) \frac{4.598}{1.072 \times 2000} - 0.6 = 2.0278}
\]

\[
\dot{m} = \frac{F}{2.0278a_0} = \frac{7931}{2.0278 \times 323.8} = 12.078 \text{ kg/s}
\]

d) The general flow rate expression is:

\[
\dot{m} = \frac{m \Gamma \rho A}{\sqrt{R T_t}} \quad (8)
\]

We have:

\[
\Gamma = \sqrt{\frac{2}{\gamma+1}} (\gamma+1)^{\frac{1}{2(\gamma-1)}} = 0.6339 \quad (9)
\]

\[
R = \frac{8.314}{0.0289} = 287 \text{ J/kg*K} \quad (10)
\]

Applying this at the choked stations 4 and 7:

\[
A_4 = \frac{\dot{m} \sqrt{R T_{t4}}}{\Gamma P_{t4}} \quad (11)
\]

\[
A_4 = 12.08 \times \frac{\sqrt{287 \times 1200}}{0.6339 \times 7.017 \times 10^5} = 0.01594 \text{ m}^2
\]

\[
A_7 = \frac{\dot{m} \sqrt{R T_{t7}}}{\Gamma P_{t7}} \quad (12)
\]

\[
A_7 = 12.08 \times \frac{\sqrt{287 \times 920.2}}{0.6339 \times 2.77 \times 10^5} = 0.03535 \text{ m}^2
\]

\[
D_7 = \frac{4}{\pi} A_7 = 0.2122 \text{ m}^2 \quad (13)
\]

For station 2:

\[
\dot{m}_2 = M_2 \left( \frac{\gamma+1}{\gamma} \right)^{\frac{\gamma+1}{2(\gamma-1)}} = 0.4 \left( \frac{1.2}{1+0.2+0.4} \right)^3 = 0.6289 \quad (14)
\]

\[
A_2 = \frac{\dot{m}_2 \sqrt{R T_{t0}}}{\Gamma P_{t0}} \times 12.08 \times \frac{\sqrt{287 \times 279.8}}{0.6289 \times 0.6339 \times 6.166 \times 10^5} = 0.1385 \text{ m}^2 \quad (15)
\]

\[
D_2 = \frac{4}{\pi} A_2 = 0.420 \text{ m} \quad (16)
\]
e) The combustion energy balance gives:

\[ f = \frac{c_p(T_i - T_f)}{h} \]  \hspace{1cm} (17)

\[ f = \frac{1005(1200 - 559.7)}{43 \times 10^6} = 0.01497 \]

Therefore:

\[ m_f = f \dot{m} = 0.01497 \times 12.08 = 0.1808 \text{ kg/s} \]  \hspace{1cm} (18)

The specific impulse is:

\[ I = \frac{F}{m_f g} = \frac{7.931}{0.1808 \times 9.8} = 4.477 \text{ s} \]  \hspace{1cm} (19)

f) The flight speed is \( u_0 = M_0 a_0 = 0.6 \times 323.3 = 194.3 \text{ m/s} \). The exhaust velocity is then:

\[ u_e = u_0 + \frac{F}{\dot{m}} = 194.3 + \frac{7931}{12.08} = 850.9 \text{ m/s} \]  \hspace{1cm} (20)

The propulsive efficiency is then:

\[ \eta_p = \frac{2 u_0}{u_0 + u_e} = \frac{2 + 194.3}{194.3 + 850.9} = 0.3718 \]  \hspace{1cm} (21)

This result is not a very high propulsive efficiency.

The overall efficiency follows from the specific impulse:

\[ \eta_{ov} = \frac{g u_0 f}{\bar{n}} = \frac{9.8 + 194.3}{43 \times 10^6} \times 4477 = 0.1983 \]  \hspace{1cm} (22)

The thermodynamic efficiency is then:

\[ \eta_{th} = \frac{\eta_{ov}}{\eta_p} = 0.5331 \]  \hspace{1cm} (23)

Note: The thermodynamic efficiency can also be calculated directly using equation (24):

\[ \eta_{th} = \frac{1}{\bar{m} f (u_e^2 - u_0^2)} \]  \hspace{1cm} (24)

**Compressor Working Line**

When conditions change, the nondimensional flow \( \bar{m}_2 \) and the compressor ratios \( \tau_c, \pi_c \) both change, but they do so in a coordinated way. As explained in lecture 19, we must have:

\[ \bar{m}_2(M_2) = \frac{A_4}{A_2} \pi_c \cdot \frac{1 - \tau_f}{\pi_c \sqrt{\frac{\gamma - 1}{\gamma - 1}}} \]  \hspace{1cm} (25)

Where \( \tau_f \) remains constant. Often \( \bar{m}_2 \) is reported as the relative flow, normalized by its design value:

\[ \bar{m}_{2D} = \frac{A_4}{A_2} (\pi_c)_D \cdot \frac{1 - \tau_f}{\pi_c \sqrt{\frac{\gamma - 1}{\gamma - 1}}} \]  \hspace{1cm} (26)

Dividing equations (25)/(26):
\[ \frac{m_2}{m_{2D}} = \frac{\pi_c}{\pi_{cD}} \sqrt{\frac{\gamma - 1}{\gamma - 1}} \]  
(27)

Some values are tabulated below, using our result \( \pi_{cD} = 11.32 \):

<table>
<thead>
<tr>
<th>( \pi_c )</th>
<th>4</th>
<th>6</th>
<th>8</th>
<th>10</th>
<th>11.32</th>
<th>12</th>
<th>14</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{m_2}{m_{2D}} )</td>
<td>0.5070</td>
<td>0.6484</td>
<td>0.7847</td>
<td>0.9159</td>
<td>1.0000</td>
<td>1.0426</td>
<td>1.1659</td>
</tr>
</tbody>
</table>

Concept Questions

1) If \( F \) is doubled, at the same \( M_0, T_0, M_2, T_{t4} \), we will still have the same \( \tau_c, \tau_t, \pi_c, \pi_t, m_2 \) and the same \( T_{t3}, T_{t4}, T_{C5}, P_{t3}, P_{t4}, P_{t5}, I, \eta_p, \eta_{th}, \eta_{ov} \). But the mass flow rate would be doubled, as would the flow areas \( A_2, A_4, A_7 \).

2) If \( T_{t4} \) is raised to 1500 K, just about everything changes. We would get more thrust per unit flow \( \frac{F}{m a_0} \), and hence less flow \( m \) and smaller cross-sections. The changes in \( I \) and in the efficiencies are less clear. The thermodynamic efficiency is that of a Brayton cycle with a pressure ratio \( (\theta_0 \tau_c) \frac{\gamma}{\gamma - 1} \), and so \( \eta_{th} = 1 - \frac{1}{\theta_0 \tau_c} = 1 - \frac{1}{\sqrt{\theta_t}} \). This means a higher \( \eta_{th} \) when \( T_{t4} \) increases. But, since a higher \( \theta_t \)
implies a higher $\frac{F}{ma_0}$, the propulsive efficiency $\eta_p$ will be less, $\eta_p = \frac{2}{2 + \left(\frac{F}{ma_0}\right)}$. The product of the two $
abla$ also turns out to be less at higher $M_0$, although this is more difficult to see.

So increasing $T_{t4}$ gives more thrust, hence a smaller engine, but at the cost of higher fuel consumption.