**Transient Heating of a Slab**

Typical problem: Uncooled throat of a solid propellant rocket

Inner layer retards heat flux to the heat sink. Heat sink’s T gradually rises during firing (60-200 sec). Peak T of heat sink to remain below matl. limit. Back T of heat sink to remain below weakening point for structure.

Prototype 1-D problem:

Can be solved exactly, or can do transient 1-D numerical computation. But it is useful to look at basic issues first.

Thermal conductance of B.L. = \( h_g \)

Thermal conductance of front layer = \( \frac{k_1}{\delta_1} \)

Thermal conductance of layer \( i \) = \( \frac{k_i}{\delta_i} \)  
(\( \delta_i \) = thickness, \( k_i \) = thermal conductivity)
Want layer 1 to have \( \frac{k_1}{\delta_1} \ll h_g \) to protect the rest.

(Say, porous, Oriented graphyte, \( k_1 = 1 \text{ W/m/K} \), \( \delta_1 = 3 \text{ mm} \)) \( k_1 = 330 \text{ W/m}^2\text{K} \) compared to \( h_g \sim 50,000 \text{ W/m}^2\text{K} \), so OK here).

Also, from governing equation
\[
\rho c \frac{\partial T}{\partial t} = k \frac{\partial^2 T}{\partial x^2} \rightarrow \frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2}
\]
\( (\alpha = \frac{k}{\rho c}, \text{ thermal diffusivity, m}^2/\text{s}) \)

we see that
\[
x^2 \sim \alpha t, \text{ or } x - \sqrt{\alpha t}, \text{ or } t - \frac{x^2}{\alpha}.
\]

So the layer 1 will "adapt" to its boundary conditions in a time \( t \sim \frac{\delta_1^2}{\alpha_1} \).

Say, \( c = 710 \text{ J/Kg K} \) and \( \rho = 1100 \text{ Kg/m}^3 \) (\( \frac{1}{2} \) solid graphyte),

so \( \alpha = \frac{1}{710 \times 1100} = 1.3 \times 10^{-6} \text{ m}^2/\text{s} \).

The layer "adapts" in \( t - \frac{\left(3 \times 10^{-3}\right)^2}{1.3 \times 10^{-6}} = 7.0 \text{ sec} \) (more like \( \frac{\delta_1^2}{4\alpha} = 1.8 \text{ sec} \)).

⇒ Treat front layer quasi-statically, i.e., responding instantly to changes in heat flux:

\[
k_1 \frac{T_{wh_i} - T_{wc_i}}{\delta_1} = q(t)
\]

This also means we can lump the thermal resistances of BL and 1st layer in series:

\[
\frac{1}{(h_g)_{\text{eff}}} = \frac{1}{h_g} + \frac{\delta_1}{k_1}
\]
and since \( \frac{k_1}{\delta_1} \ll h_g \),

\[
(h_g)_{\text{eff}} - \frac{k_1}{\delta_1} \ll h_g
\]

For layer 2 (the heat sink), \( k_2 \) is high (metal) and \((h_g)_{\text{eff}}\) is now small (thanks to 1st layer) so, very likely,

\[
\frac{k_2}{\delta_2} \gg (h_g)_{\text{eff}}
\]

(For instance, say Copper, \( k_2 = 360 \frac{W}{mK} \), with \( \delta_2 = 2 \text{ cm} \). We now have

\[
(h_g)_{\text{eff}} = \frac{k_2}{\delta_2} = 350 \frac{W}{m^2K}, \text{ but } \frac{k_2}{\delta_2} = 36,000 \frac{W}{m^2K}, \text{ so indeed, } \frac{k_2}{\delta_2} \gg (h_g)_{\text{eff}}.
\]

Under these conditions, the heat sink is being “trickle charged” through the high thermal resistance of layer 1. Most likely, heat has time to redistribute internally, so that \( T_2 \) is nearly uniform across the layer. We can then write a lumped equation.

\[
\rho_2 c_2 \delta_2 \frac{dT_2}{dt} = q = (h_g)_{\text{eff}} (T_{aw} - T_2) = \frac{k_1}{\delta_1} (T_{aw} - T_2)
\]

Define

\[
\tau = \frac{\rho_2 c_2 \delta_1 \delta_2}{k_1} \quad \tau \frac{dT_2}{dt} + T_2 = T_{aw} \quad (T_2 (0) = T_0)
\]

\[
T_2 = T_{aw} - (T_{aw} - T_0) e^{-\frac{t}{\tau}}
\]

For our example, say \( \rho_2 = 8900 \text{ Kg/m}^3 \) (Copper), \( c_2 = 430 \frac{J}{KgK} \), \( \delta_2 = 2 \text{ cm} \)

\[
\tau = \frac{8900 \times 430 \times 3 \times 10^{-3} \times 2 \times 10^{-2}}{1} = 230 \text{ sec}
\]
This is comfortable. Suppose $T_{aw} = 3300K$, $T_0 = 300K$, and we fire for 120 sec:

$$T_2(120) = 3300 - 3000e^{\frac{120}{230}} = 1520K$$

(989) May need 4 cm

which is still (OK) for Copper (melts at 1360K, but no stress bearing, so can go to ~900. Also OK for steel on Carbon str member).

NOTE:

$$\frac{\delta_2^2}{4\alpha_2} = \frac{(0.02)^2}{4 \times 9.4 \times 10^{-5}} = 1.1 \text{ sec, so, indeed, layer 2 "adapts" quickly to B.C.'s}$$

$$\to \text{ uniform } k_2 \rho_2 c_2 = \frac{360}{8900 \times 430} = 9.4 \times 10^{-5} \text{ m}^2 / \text{s}.$$

A More Exact Solution

Consider $T_{aw}$ “turned on” at $t=0$. The B.L. has a film coefficient $h_g$, and the first layer has $\delta_1$, $k_1$, so that $(h_g)_{eff} = \frac{h_g}{1 + h_g \frac{\delta_1}{\delta_1}}$. Layer 2 has thickness $\delta_2$, and has $k_2$, $\rho_2$, $\sigma_2$, $\alpha_2$. The back is insulated.

Then one can prove that layer 2 has a temperature distribution
\[
\frac{T_{aw} - T_2(x,t)}{T_{aw} - T_0} = \sum_{n=1}^{\infty} a_n e^{-\lambda_n^2 \frac{\alpha_2}{\delta_2} t} \cos \left( \frac{\delta_2 - x}{\delta_2} \lambda_n \right)
\]

where \( a_n = \frac{2 \sin \lambda_n}{\lambda_n + \sin \lambda_n \cos \lambda_n} \)

and \( \lambda_n \) (n=1,2,...) are the roots of

\[
\lambda_n \tan \lambda_n = \left( \frac{h_1}{\delta_2} \right)_{eff} \frac{\delta_2}{k_2} = \frac{k_1 \delta_2}{k_2 \delta_1}
\]

Graphically,

For small \( \Delta = \frac{k_1 \delta_2}{k_2 \delta_1} \), small \( \lambda_1 \), so \( \tan \lambda_1 = \lambda_1 \), so

\[
\lambda_1^2 = \Delta \lambda_1 = \sqrt{\Delta} = \frac{k_1 \delta_2}{k_2 \delta_1}
\]

and also \( a_1 = 1 \)

\[
\lambda_1^2 \frac{\alpha_2}{\delta_2} = \frac{k_1}{\delta_1} \left( \frac{\delta_2}{\rho_2 c_2} \right) \frac{\lambda_1^2}{\delta_2} = \frac{k_1}{\rho_2 c_2 \delta_1 \delta_2} = \tau
\]

from before

So, leading term is then

\[
\frac{T_{aw} - T_2(x,t)}{T_{aw} - T_0} = e^{-\frac{t}{\tau}} \cos \left( \frac{\delta_2 - x}{\delta_2} \lambda_1 \right)
\]
which is what we found before. The other terms are much smaller, except at very small time.


A key concept is ablative materials. They contain a C-based homogeneous matl. embedded in reinforcing fibres of strong (anisotropic) C. Best is C/C, strong expensive fibre since nozzle can get to 3600 K, can be 2D or 3D. Also good is C or Kelvin (Aramid) fibres +phenolic plastic resins (for large nozzles)

For the shuttle RSRM, the throat insert (C cloth phenolic) regresses ~ 1 inch/120 sec, and the char depth is ~ 0.5° inch/120 s.
Film Cooling of Rockets

For application of data on slot-injected films, we need to define the initial film thickness \( s \), velocity \( u_F \), density \( \rho_F \), or at least mass flux \( u_F \rho_F \).

Assume we know the flow rates \( \dot{m}_c \) and \( \dot{m}_F \), where \( \dot{m}_c \) is the “core” flow and \( \dot{m}_F \) the “film” flow. We also know the fully-burnt temperatures and molecular weights \( (T_c, T_F ; M_c, M_F) \).

The areas occupied at the “fully burnt” section are not known; let them be \( A_c, A_F \).

From continuity,

\[
\dot{u}_c A_c = \frac{\dot{m}_c}{\rho_c} = \frac{\dot{m}_c}{P} \frac{R}{M_c} T_c \quad \quad \quad (1)
\]

\( P = P_c \) is common to both.
and the total cross-section is known:

\[ A_c + A_F = A \]  \hspace{1cm} (3)

We need some additional information to find \( u_F \). The two momentum equations are (neglecting friction):

\[
\begin{align*}
\rho u_c \frac{du_c}{dx} + \frac{dP}{dx} &= 0 \\
\rho u_F \frac{du_F}{dx} + \frac{dP}{dx} &= 0
\end{align*}
\]

Both, \( \rho_c \) and \( \rho_F \), have been evolving as drops evaporate and burn. We make now the approximation of assuming their ratio to remain constant (equal to the fully-burnt value). Then (4) integrates to

\[ \frac{u_F^2}{u_c^2} = \frac{\rho_c}{\rho_F} \]

Substitute into the ratio (2)/(1)

\[ \frac{\rho_F u_F A_F}{\rho_U U_c A_c} = \frac{\dot{m}_F}{\dot{m}_c} \rightarrow \rho_F \sqrt{\frac{\rho_c}{\rho_F}} \frac{A_F}{A_c} = \frac{\dot{m}_F}{\dot{m}_c} \]

or

\[ A_F = \frac{\dot{m}_F}{\dot{m}_c} \sqrt{\frac{\rho_c}{\rho_F}} \]

and also

\[ \frac{\rho_F U_F}{\rho_U U_c} = \sqrt{\frac{\rho_c}{\rho_F}} \]

This last ratio \( \left( \frac{\rho_F U_F}{\rho_U U_c} \right) \) is called the “film cooling parameter”, \( M_F \) :
The film thickness \( s \) (at complete burn up) follows from

\[
A_F = \pi \left[ D^2 - \left( D - 2s \right)^2 \right]
\]

\[
A_c = \pi \left( D - 2s \right)^2
\]

\[
s = \frac{D A_F}{4 A_c} = \frac{D \dot{m}_F}{4 \dot{m}_c} \sqrt{\frac{\rho_F}{\rho_c}}
\]  

From Rosenhow & Hartnett, Chapter 17-B, we characterize film cooling by the change it induces to the driving temperature \( T_{aw} \) for heat flow. In the absence of a film, \( T_{aw}^0 = T_c \left( 1 + r \frac{\gamma - 1}{2} M_c^2 \right) \), and we calculate \( (q_{aw})_{NoFilm} = h_g \left( T_{aw}^0 - T_w \right) \). The film changes \( T_{aw} \) to \( T_{aw}^F \) (lower, presumably). The lowest we could \( T_{aw} \) to get is \( T_F \), so we define a film cooling efficiency

\[
\eta = \frac{T_{aw}^0 - T_{aw}^F}{T_{aw} - T_F}
\]  

Limits: \( \begin{cases} 
\eta = 0 \quad \text{if} \quad T_{aw}^F = T_{aw}^0 \quad \text{(no effect)} \\
\eta = 1 \quad \text{if} \quad T_{aw}^F = T_F \quad \text{(maximum effect)}
\end{cases} \)

If we can predict \( \eta \), then

\[
T_{aw}^F = T_{aw}^0 - \eta \left( T_{aw}^0 - T_F \right)
\]

and then

\[
q_{aw} = h_g \left( T_{aw} - T_w \right)
\]  

where \( h_g \) is computed as if there were no film. To predict \( \eta \), we first computes the parameter
\[ \zeta = \frac{x}{M_F S \left( \frac{Re_F \mu_F}{\mu_c} \right)^{\frac{1}{4}}} \]  

(13)

where \( x \) is the distance downstream of the film injection (here we assume this is from the burn-out section), and

\[ Re_F = \frac{\rho u_F s}{\mu_F} \]  

(14)

and \( \rho u_F = M_F (\gamma u_c) \), from before

From \( \zeta \), there are several semi-empirical correlations for \( \eta \). A recommendation from R & H is

\[ \eta = \frac{1.9 \rho_{F}^{\frac{2}{3}}}{1 + 0.329 \left( \frac{c_{pF}}{c_{pC}} \right) \zeta^{0.8}} \]

(15)

(or \( \eta = 1 \) if this gives >1)

which is supported by air data of Seban.

**Example**

Say \( \frac{T_F}{T_c} = \frac{1}{2} \); \( M_F = 0.8 \rightarrow \frac{\rho_F}{\rho_c} = \frac{0.8}{0.5} = 1.6 \rightarrow M_F = \sqrt{1.6} = 1.265 \)

\[ \frac{m_F}{m} = 0.1 \rightarrow \frac{m_F}{m_c} = \frac{1}{9} \]

Say \( D=0.5m \) \( x_t - x_{compl.comb} = 0.5 \ m \)

\[ P=70 \ atm=7.09 \times 10^6 \ N/m^2 \]
\[ T_c = 3200K \]
\[ M_c = 20 \ g/mol; \gamma_c = 1.2 \]

\[ \rho_c = \frac{7.09 \times 10^6 \times 0.020}{8.314 \times 3200} = 5.33 \text{Kg}/\text{m}^3; \mu_c = 8.53 \text{Kg}/\text{m}^3 \]

\[ M_c = 0.2 \]
\[
\begin{align*}
u_c &= 0.2 \sqrt{\frac{1.2 \times 8.314}{0.02} \times 3200} = 253 \text{ m/s} \\
u_f &= 253 \sqrt{\frac{1}{1.6}} = 200 \text{ m/s} \\
\text{Say } \mu_f &= 2 \times 10^{-5} \text{ Kg/m/s} \rightarrow \Re_f = \frac{8.53 \times 200 \times s}{2 \times 10^{-5}} = 8.53 \times 10^7 \text{ s} \\
\Re_f &= 9.37 \times 10^5 \\
s &= \frac{D m_f}{4 m_c} \sqrt{\frac{\mu_f}{\rho_f}} = \frac{0.5}{4} \times \frac{1}{\frac{9}{16}} \sqrt{\frac{1}{1.6}} = 0.0110 \text{ m} \\
\text{OR } 0.0101 \\
\frac{\mu_f}{\mu_c} &= \left( \frac{T_f}{T_c} \right)^{0.6} = 0.5^{0.6} = 0.660 \\
\zeta &= \frac{0.5}{1.265 \times 0.0110} \left( \frac{9.37 \times 10^5 \times 0.660}{8.51 \times 10^4} \right)^{\frac{1}{4}} = 1.282 \\
(25.74) \\
e_{p,r} &= \frac{\mu_f}{\mu_c} = 0.8 \quad \text{(say, } r_F = r_C), \quad P_F = 0.8 \\
\eta &= \frac{1.9 \times 0.8^{2/3}}{1 + 0.329 \times 0.8 \times 1.282^{0.8}} \approx 0.368 \quad 0.361 \\
(25.74)^{0.8} \\
\text{So, this offers perfect film cooling, meaning} \\
T_{sw}^F &= T_F = \frac{T_c}{2} = 1600 \text{ K} \\
(3200-0.361(3200-700)=2296 \text{ K}) \\
\text{If the wall is made of Cu, and is at } T_w = 700 \text{K, the reduction in heat flow is} \\
\frac{q_w^F}{q_w^0} &= \frac{1600 - 700}{3200 - 700} = 0.360 \\
\left( \frac{2296 - 700}{3200 - 700} = 0.638 \right) \\
\text{which can be decisive.}
\end{align*}
\]
(This example shows one could get good film cooling with much less than 10% flow in the film, maybe with only 2%).