Ideal Nozzle Flow with No Separation (1-D)

- Quasi 1-D (slender) approximation
- Ideal gas assumed

\[ F = m u_e + (P_e - P_a) A_e \]

\[ C_F = \frac{F}{P_c A_t} \]

Optimum expansion: \( P_e = P_a \)

- For less \( \frac{A_e}{A_t} \), \( P_e > P_a \), could derive more forward push by additional expansion
For more $\frac{A_e}{A_t}$, $P_e < P_a$, and the extra pressure forces are a \textit{suction}, backwards.

Compute $m = \rho u A$ at \textit{sonic} throat:

$$m = \rho_c \left(\frac{2}{\gamma + 1}\right)^{1/2} \sqrt{\gamma R_t c \left(\frac{2}{\gamma + 1}\right) A_t} = \sqrt{\gamma \frac{2}{\gamma + 1} \frac{P_c A_t}{\sqrt{R_t T_c}}} \quad ; \quad R_g = \frac{R}{M}$$

call $\Gamma \equiv \frac{2}{3}$

call $c^* = \frac{\sqrt{R_t T_c}}{\Gamma(\gamma)}$ ("characteristic velocity") \rightarrow $m = \frac{P_c A_t}{c^*}$

Can express $u_e$, $P_e$, $A_e$, etc in terms of either $M_e$ or $\left(\frac{P_e}{P_c}\right)$ or $\frac{A_e}{A_t}$:

$$\frac{P_e}{P_c} = \frac{1}{\left(1 + \frac{\gamma - 1}{2} M_e^2\right)^{\gamma/2}} \quad ; \quad \left(\frac{P_e}{P_c}\right)^{\frac{\gamma + 1}{2(\gamma - 1)}}$$

$$\frac{A_e}{A_t} = \left(\frac{P_e}{P_c}\right) \left(\frac{u_t}{u_e}\right) = \left(\frac{P_e}{P_c}\right) \frac{1}{M_e} \sqrt{\frac{T_e}{T_c}} = \frac{1 + \frac{\gamma - 1}{2} M_e^2}{\gamma + 1} \frac{\frac{1}{2} \frac{1}{\gamma - 1}}$$

$$\frac{P_e}{P_c} = \left(\frac{T_e}{T_c}\right)^{\gamma/2}$$

and $\frac{T_e}{T_c} = \frac{1}{1 + \frac{\gamma - 1}{2} M_e^2}$

Because $c_p T_e + \frac{u_e^2}{2} = c_p T_c \rightarrow \gamma \frac{R_g T_e}{\gamma - 1} + \frac{M_e^2}{2} \gamma R T_e = \gamma \frac{R_g T_c}{\gamma - 1}$
\[ C_F = \frac{m}{P_c A_t} u_e + \left( \frac{P_e - P_a}{P_c} \right) \frac{A_e}{A_t} = \frac{u_e}{c^*} + \left( \frac{P_e - P_a}{P_c} \right) \frac{A_e}{A_t} \]

\[ u_e \frac{c^*}{c} = \frac{M_e}{\sqrt{R_g R_c}} \frac{\frac{\gamma c}{1 + \frac{\gamma - 1}{2} M_e^2}}{\frac{\sqrt{R_g R_c}}{\Gamma}} = \gamma \left( \frac{2}{\gamma + 1} \right)^{\frac{\gamma + 1}{2(\gamma - 1)}} \frac{M_e}{\sqrt{1 + \frac{\gamma - 1}{2} M_e^2}} \]

In vacuum,

\((P_a = 0)\)

\[ C_{Fv} = u_e \frac{c^*}{c} + \frac{P_e}{P_c} A_e = \left( \frac{2}{\gamma + 1} \right)^{\frac{\gamma + 1}{2(\gamma - 1)}} \frac{\gamma M_e}{\sqrt{1 + \frac{\gamma - 1}{2} M_e^2}} + \frac{1}{M_e} \frac{1 + \frac{\gamma - 1}{2} M_e^2}{M_e} \left( \frac{\gamma + 1}{2(\gamma - 1)} \right)^{\frac{\gamma + 1}{2(\gamma - 1)}} \]

\[
(C_F)_{vac} = \left( \frac{2}{\gamma + 1} \right)^{\frac{\gamma + 1}{2(\gamma - 1)}} \frac{\gamma M_e + \frac{1}{M_e}}{\sqrt{1 + \frac{\gamma - 1}{2} M_e^2}}
\]

and otherwise,
\[ C_F = C_{Fv} - \left( \frac{A_e}{A_t} \right) \frac{p_a}{p_c} \]

Note:

For \( p_e = p_a \),

\[ (C_F)_{Matched} = \frac{u_e}{c^*} = \left( \frac{2}{\gamma + 1} \right)^{\frac{\gamma + 1}{2(\gamma - 1)}} \frac{\gamma M_e}{\sqrt{1 + \frac{\gamma - 1}{2} M_e^2}} \]

For \( p_e = p_a = 0 \),

\[ (C_F)_{Max,Vac} = \gamma \left( \frac{2}{\gamma - 1} \right)^{\frac{\gamma + 1}{2(\gamma - 1)}} \]

**Choice of Optimum Expansion For a Rocket Flying Through an Atmosphere (\( p_a \) varying)**

The thrust coefficient \( C_F = \frac{F}{P_c A_t} \) was derived in class in the form

\[ C_F = C_{Fvac} - \frac{p_a}{p_c} \left( \frac{A_e}{A_t} \right) \]  

(1)

\[ C_{Fvac} = \left( \frac{2}{\gamma + 1} \right)^{\frac{\gamma + 1}{2(\gamma - 1)}} \frac{\gamma M_e + 1/\sqrt{M_e}}{\sqrt{1 + \frac{\gamma - 1}{2} M_e^2}} \]  

(2)

and we also found

\[ \frac{A_e}{A_t} = \frac{1}{M_e} \left( \frac{1 + \frac{\gamma - 1}{2} M_e^2}{\frac{\gamma + 1}{2}} \right)^{\frac{\gamma + 1}{2(\gamma - 1)}} \]  

(3)

The thrust-derived velocity increment is

\[ \Delta V_F = \int_0^{t_b} \frac{F}{m} dt = P_c A_t \int_0^{t_b} \frac{C_F}{m} dt \]  

(4)
where \( C_F = C_F(t) \) due only to the variation of \( P_a \) in (1), while \( m = m(t) \) because of mass burnout. The quantities \( C_{F\text{vac}} \) and \( \frac{A_e}{A_t} \) depend on \( M_e \) (or nozzle geometry), but are time-invariant. Substituting (1), (2) and (3) into (4),

\[
\Delta V_F = P_c A_t \left[ C_{F\text{vac}} \int_0^{t_b} \frac{dt}{m} - \left( \frac{A_e}{A_t} \right) \int_0^{t_b} \frac{P_a}{P_c} \frac{dt}{m} \right]
\]

or

\[
\frac{\Delta V_F}{P_c A_t} \int_0^{t_b} \frac{dt}{m} = C_{F\text{vac}} \int_0^{t_b} \frac{P_a}{P_c} \frac{dt}{m} - \left( \frac{A_e}{A_t} \right) \int_0^{t_b} \frac{dt}{m} \tag{5}
\]

We now make the approximation that the trajectory will change little when we vary \( M_e \) (and hence \( C_{F\text{vac}}, \frac{A_e}{A_t} \)). We can then regard the time integrals in (5) as fixed quantities while we optimize \( M_e \). Define the non-dimensional variables

\[
\nu = \frac{\Delta V_F}{P_c A_t} \int_0^{t_b} \frac{dt}{m} ; \quad \rho = \int_0^{t_b} \frac{P_a}{P_c} \frac{dt}{m} \int_0^{t_b} \frac{dt}{m} \tag{6}
\]

so that (5) becomes

\[
\nu = C_{F\text{vac}} (M_e) - \rho \left( \frac{A_e}{A_t} \right) (M_e) \tag{7}
\]

and we can now differentiate \( \nu \) w.r.t \( M_e \) (holding \( \rho=\text{const.} \))

\[
\frac{\partial \nu}{\partial M_e} = \frac{\partial C_{F\text{vac}}}{\partial M_e} - \rho \left( \frac{\partial \left( \frac{A_e}{A_t} \right)}{\partial M_e} \right) = 0 \tag{8}
\]

From (2) and (3), the factor \( \left( \frac{2}{\gamma + 1} \right)^{\frac{\gamma + 1}{2|\gamma - 1|}} \) appears in both terms of (8) and can be ignored. We then have
\[ \frac{\partial}{\partial M_e} \left( \frac{\gamma M_e + 1/M_e}{\sqrt{1 + \frac{\gamma - 1}{2} M_e^2}} \right) = P \frac{\partial}{\partial M_e} \left[ \left( 1 + \frac{\gamma - 1}{2} M_e^2 \right)^{2(\gamma - 1)} \right] \]

\[ \frac{\gamma - 1/M_e^2}{\sqrt{1 + \frac{\gamma - 1}{2} M_e^2}} \left( \gamma M_e + 1 \right) \frac{1}{\sqrt{2}} \left( 1 + \frac{\gamma - 1}{2} M_e^2 \right)^{\gamma + 1} = p \left( 1 + \frac{\gamma - 1}{2} M_e^2 \right)^{\gamma - 1/2} \left[ \frac{\gamma + 1}{2} - \frac{\gamma - 1}{2} M_e^2 \right] \\
\]

Multiply times \( \left( 1 + \frac{\gamma - 1}{2} M_e^2 \right)^{\gamma/2} \), and note that \( \frac{\gamma + 1}{2} + \frac{1}{\gamma - 1} = \frac{\gamma}{\gamma - 1} \)

\[ \left( \frac{\gamma - 1}{M_e^2} \right) \left( 1 + \frac{\gamma - 1}{2} M_e^2 \right) - \frac{\gamma - 1}{2} (\gamma M_e^2 + 1) = p \frac{\left( 1 + \frac{\gamma - 1}{2} M_e^2 \right)^{\gamma - 1/2}}{M_e^2} \left[ (\gamma + 1) M_e^2 - \left( 1 + \frac{\gamma - 1}{2} M_e^2 \right) \right] \\
\]

Expand & simplify

\[ \gamma + \frac{\gamma (\gamma - 1)}{2} M_e^2 - \frac{1}{M_e^2} - \frac{\gamma - 1}{2} (\gamma M_e^2 - \frac{\gamma - 1}{2} M_e^2) = p \frac{\left( 1 + \frac{\gamma - 1}{2} M_e^2 \right)^{\gamma - 1/2}}{M_e^2} \left( M_e^2 - 1 \right) \\
\]

\[ 1 - \frac{1}{M_e^2} = \frac{M_e^2 - 1}{M_e^2} \\
\]

Cancel the factor \( \frac{M_e^2 - 1}{M_e^2} \) \( \left( M_e = 1 \text{ is clearly not an optimum!} \right) \)

\[ 1 = p \left( 1 + \frac{\gamma - 1}{2} M_e^2 \right)^{\gamma/2 - 1} \]

or
\[ 1 + \frac{\gamma - 1}{2} M_e^2 = \left( \frac{1}{p} \right)^{\frac{\gamma - 1}{\gamma}} \]

\[ M_{e_{OPT}} = \sqrt[\gamma - 1]{\frac{2}{\gamma - 1} \left( \frac{1}{\rho} \right)^{\frac{\gamma - 1}{\gamma}} - 1} \]  

(9)

Notice that the exit pressure is given by

\[ \frac{P_e}{P_c} = 1 \left( \frac{1 + \frac{\gamma - 1}{2} M_e^2}{1 + \frac{\gamma - 1}{2} M_e^2} \right)^{\frac{1}{\gamma - 1}} \]  

(10)

and so the optimum exit pressure turn out to be

\[ \frac{\left( P_e \right)_{OPT}}{P_c} = p \]  

(11)

However, if \( p < 0.4 \frac{P_{ao}}{P_c} \), this would imply \( P_e < 0.4 P_{ao} \), and there would be flow separation at the highest \( P_a \) (on the ground). To avoid this, the optimality condition must be amended to

\[ \frac{\left( P_e \right)_{OPT}}{P_c} = \text{Greater of } \left\{ p, 0.4 \frac{P_{ao}}{P_c} \right\} \]  

(12)

with a similar expression for \( M_e \):

\[ M_{e_{OPT}} = \text{Least of } \left\{ \sqrt[\gamma - 1]{\frac{2}{\gamma - 1} \left( \frac{1}{\rho} \right)^{\frac{\gamma - 1}{\gamma}} - 1}, \sqrt[\gamma - 1]{\frac{2}{\gamma - 1} \left( \frac{2.5 P_e}{P_{ao}} \right)^{\frac{\gamma - 1}{\gamma}} - 1} \right\} \]  

(13)

The limiting condition in which the whole burn occurs at \( P_{ao} \) is simple.

We then obtain

\[ p = \frac{\int_0^{t_b} P_{ao} \frac{dt}{m}}{\frac{\int_0^{t_b} \frac{dt}{m}}{m}} = \frac{P_{ao}}{P_c} \]  

(14)
and the optimality condition (12) yields $(P_e)_{OPT} = P_{ao}$, i.e., the nozzle should be pressure-matched, as expected.

As more and more of the burn shifts to higher altitudes, $p$ decreases from $\frac{P_{ao}}{P_c}$. As long as it still remains above $0.4\frac{P_{ao}}{P_c}$, equation (11) gives some intermediate optimum design, and if $p$ drops below $0.4\frac{P_{ao}}{P_c}$, the nozzle should be designed to be on the verge of separation on the ground.

**Nozzle Flow Separation Effects**

Rule of thumb (to be explored later):

Flow separates at the point in the nozzle where

$$P \geq 0.4P_a$$ (Summerfield criterion)

So, if $P_e > 0.4P_a$ (even if $P_e < P_a$), no separation

After separation, roughly parallel flow, at $P = P_a$ (no strong $p$ gradients in “dead water” region to turn flow).

So zero thrust contribution $\rightarrow$ Performance with separation at that of a nozzle with exit pressure $P_e' = 0.4P_a$

So,

(a) $P_a < \frac{P_e(\text{full nozzle})}{0.4}$,

$$C_F = C_{F_{vac}} - \frac{P_a}{P_o} \frac{A_e}{A_t}$$

$$f(M_e) \quad g(M_e)$$
(b) \( P_a > \frac{P_e \text{(full nozzle)}}{0.4} \),

\[
M'_e = M \left( \frac{P_e'}{P_a} = 0.4 P_a \right)
\]

then

\[
C_F = C_{F\text{vac}} \left( M'_e \right) - \frac{P_a}{P_0} \frac{A_e}{A_t}
\]