Idealized Direct GTO Injection
(GTO = Geosynchronous Transfer Orbit)

Assumptions:

- Ignore drag and "gravity" losses
- Assume impulsive burns (instantaneous impulse delivery)
- Assume all elevations $\alpha > 0$ at launch are acceptable

Launch is from a latitude $L$, directed due East for maximum use of Earth's rotation. The Eastward added velocity due to rotation is then

$$V_R = \Omega R_E \cos L = 463 \cos L \text{ (m/s)} \quad (1)$$

If the launch elevation is $\alpha$, and the desired velocity after the first burn is $V_1$, the rocket must supply a velocity increment

$$\Delta V_1 = \sqrt{V_1^2 + V_R^2 - 2 V_1 V_R \cos \alpha} \quad (2)$$

The trajectory will then lie in a plane LOI through the Earth's center which contains the local E-W line. In order to be able to perform the plane change to the equatorial plane at GEO, we select the elevation $\alpha$ such as to place the apogee of the transfer orbit (GTO) at the GEO radius $R_{GEO} = \left(\frac{\mu T^2}{4\pi^2}\right)^{1/3} = \frac{42,200}{T = 24 \text{ hr}, \mu = 3.986 \times 10^{14} \text{ m}^3/\text{s}^2}$
Since OL is perpendicular to OI, the view in the plane of the orbit is:

The polar equation of the trajectory is 
\[ r = \frac{p}{1 + e \cos \theta}, > 0 \]

In our case \( p = R_E \) (corresponding to \( \theta = \frac{\pi}{2} \)). The elevation is given by
\[
\tan \alpha = \left( \frac{dr}{r \, d\theta} \right)_{\theta = \pi / 2} = \left( \frac{e \sin \theta}{(1 + e \cos \theta)^2} \right)_{\theta = \pi / 2} = e
\]

and, in turn, the eccentricity follows from (at \( \theta = \pi \))

\[
R_{GEO} = \frac{R_e}{1 - e} \quad \quad e = 1 - \frac{R_e}{R_{GEO}}
\]

and so \( \tan \alpha = 1 - \frac{R_e}{R_{GEO}} = 0.849 \); \( \alpha = 40.3^\circ \) (3)

The angular momentum (per unit mass) is \( h = \sqrt{\mu p} = \sqrt{\mu R_e} \).

Equating this to \( R_e V_i \cos \alpha \),

\[
V_i \cos \alpha = \frac{\mu}{\sqrt{R_e}}
\]

(i.e., the horizontal projection of the launch velocity is the local orbital speed, for any apogee radius, \( R_{GEO} \) in this case)

Combining (3) and (4),

\[
V_i = \frac{\mu}{R_e} \left[ 1 + \left( 1 - \frac{R_e}{R_{GEO}} \right)^2 \right]^{1/2}
\]

and this can now be substituted in (2):

\[
\Delta V_i = \frac{\mu}{\sqrt{R_e}} \left[ 1 + \left( 1 - \frac{R_e}{R_{GEO}} \right)^2 \right] + v_r^2 - 2v_r \sqrt{\mu R_e}
\]

\[
\Delta V_i = \sqrt{\left( \frac{\mu}{R_e} - v_r \right)^2 + \frac{\mu}{R_e} \left( 1 - \frac{R_e}{R_{GEO}} \right)^2}
\]

Upon arrival at I, there will have to be a second burn that will simultaneous accelerate the rocket to \( v_{GEO} = \frac{\mu}{\sqrt{R_{GEO}}} \), and rotate the plane to equatorial ( \( \Delta i = L \)).
The apogee velocity is \( v_{a,GTO} \), given by

\[
R_{GEO} v_{a,GTO} = (V_i \cos \alpha)R_E = \sqrt{\mu}R_E
\]  

(7)

and so \( \Delta v_a = \sqrt{v_{GEO}^2 + v_{a,GTO}^2 - 2v_{GEO}v_{a,GTO} \cos \Delta i} \)

\[
\Delta v_a = \sqrt{\frac{\mu}{R_{GEO}}} \sqrt{1 + \frac{R_E}{R_{GEO}} - 2 \frac{R_E}{R_{GEO}} \cos \Delta L}
\]  

(8)

This second burn is probably provided by the spacecraft itself, or else by the launcher's upper stage.

**IDEALIZED TWO - BURN GTO INJECTION**

One difficulty with the direct injection scheme is the fact that GEO insertion at \( I \) must occur on the first pass, because the GTO perigee is actually below the Earth's surface (see Fig. 2). Most operators prefer a temporary parking of the spacecraft in a GTO orbit which has a perigee above the ground, so as to make functional tests and adjustments prior to the final apogee burn (over a period of 2-4 weeks). A modification of the launch sequence to accommodate this is:

(1) Fire Eastwards with \( \alpha \) selected for a low apogee (– 200 km above ground) at the equatorial crossing.
(2) Fire again at equatorial crossing to raise the apogee to \( R_{GEO} \) (no plane change)
(3) At one of the apogee passes, perform the final (circularization + plane change burn).
The formulation is very similar to the previous case. The elevation $\alpha$ is now given by

$$\tan \alpha = 1 - \frac{R_E}{R_p}$$  \hspace{1cm} (9)

($R_p$ = perigee radius = $R_E + 200$ km).

This gives a very shallow trajectory, which is unrealistic; but it is a fair approximation to a real high-elevation launch, followed by a rapid rotation during the rocket firing. For $R_p - R_e = 200$ km, $\alpha = 1.74^\circ$.

Eqs. (5) and (6) still hold, with the quality $R_{\text{GEO}}$ replaced by $R_p$, and so

$$\Delta V_1 = \sqrt{\left(\frac{\mu}{R_E} - \frac{v_r}{R_p}\right)^2 + \frac{\mu}{R_E} \left(1 - \frac{R_E}{R_p}\right)^2}$$  \hspace{1cm} (10)

which is now smaller, since we are going to a much lower apogee (at $r_p$).

At this apogee (at the equatorial crossing), we have, as in Eq. (7),

$$v_a = \sqrt{\frac{\mu R_E}{R_p}}$$  \hspace{1cm} (11)

and we next need to effect a second rocket firing that will increase velocity to that for the GTO perigee:

$$v_{\text{GTO}} = \sqrt{\frac{\mu}{R_p \left(1 + \frac{2R_{\text{GEO}}}{R_p + R_{\text{GEO}}^2}\right)}}$$  \hspace{1cm} (12)
No plane change is involved yet, so

$$\Delta V_2 = \frac{\mu}{\sqrt{R_p}} \left[\frac{2R_{GEO}}{\sqrt{R_p + R_{GEO}} - \sqrt{R_p}}\right]$$

(13)

This places the spacecraft on an elliptical GTO orbit, still in the original plane, with apogee at $R_{GEO}$. The speed at this apogee is:

$$v_{a,GTO} = \frac{\mu}{\sqrt{R_{GEO}}} \frac{2R_p}{R_p + R_{GEO}}$$

(14)

and so,

$$\Delta V_a = \sqrt{v_{GEO}^2 + v_{a,GTO}^2 - 2v_{GEO} v_{a,GTO} \cos L}$$

$$\Delta V_a = \frac{\mu}{\sqrt{R_{GEO}}} + \frac{\mu}{\sqrt{R_{GEO}}} \frac{2R_p}{R_p + R_{GEO}} - 2 \frac{\mu}{\sqrt{R_p + R_{GEO}}} \frac{2R_p}{\sqrt{R_p + R_{GEO}}} \cos L$$

(15)
Some numerical comparisons

We will illustrate these $\Delta V$'s by considering launches to GEO from two different locations:

1. Near the Equator, on at the French kouron complex, and
2. From mid-latitude, as from Café Canoveral ($L = 28.5^\circ$).

(1) Equatorial Launch

Option (a): Ground to LEO (300 km), plus LEO-GEO Hohman transfer. No plane changes. Launch to the East.

\[
\Delta V = \Delta V_1 + \Delta V_2 - V_R + \Delta V_3 + \Delta V_4
\]

\[
\Delta V = (8084 - 463) + (10,151 - 7725) + (3071 - 1573)
\]

\[
= 7,621 + 2,426 + 1,498 = 11,545 \text{ m/s}
\]

Notice this is more than to Escape from mean Earth ($\Delta V = 11,200 \text{ m/s}$)

Option (b): Direct injection into GTO from ground

\[
\Delta V = \Delta V_2 + \Delta V_3
\]

\[
= (10,420 - 463) + (3071 - 1573)
\]

\[
= 9,957 + 1,498 = 11,455 \text{ m/s}
\]

(2) Launch from $L = 28.5^\circ$. Launch to East, $\nu_R = 407 \text{ m/s}$

Option (a): Direct injection to GTO, circularization + plane change at GEO. 2 firings,

\[
\Delta V = \Delta V_1 + \Delta V_2
\]

\[
= 10,070 + 2,102 = 12,172 \text{ m/s}
\]

Note the two penalizations for latitude: the elevated launch increased $\Delta V_1$, and the plane change at GEO increases $\Delta V_2$.

Option (b) Direct injection with 3 firings (LEO at 300km)

\[
\Delta V = \Delta V_1 + \Delta V_2 + \Delta V_3
\]

\[
= 7,512 + 2,605 + 1,830 = 11,947 \text{ m/s}
\]
Is it true that plane change should be all done at end of GTO?

Actually, a small turning combined with initial $\Delta V_1$ (say, from LEO) costs very little $\Delta V$ loss, even though $V$ is then large. Try splitting into a $\Delta i_1$ and $\Delta i_2 = \Delta i - \Delta i_1$

$$\Delta V_1 = \sqrt{v_c^2 + v_{\text{p,over}}^2 - 2v_c v_{\text{p,over}} \cos \Delta i_1}$$

$$\Delta V_2 = \sqrt{v_c^2 + v_{\text{p,over}}^2 - 2v_c v_{\text{p,over}} \cos (\Delta i - \Delta i_1)}$$

$$\Delta V = \Delta V_1 + \Delta V_2$$

$$\frac{d\Delta V}{d\Delta i_1} = \frac{1}{2} v_c V_{\text{p,over}} \sin \Delta i_1 - \frac{1}{2} v_c V_a \sin (\Delta i - \Delta i_1) = 0$$

$$v_c_1 = \sqrt{\frac{\mu}{R_1}}, \quad v_c_2 = \sqrt{\frac{\mu}{R_2}}, \quad v_p = \sqrt{\frac{\mu}{R_1 + R_2}}, \quad v_a = \sqrt{\frac{\mu}{R_2}}$$

Call $\rho = \frac{R_2}{R_1}$

$$\sqrt{\frac{2\rho}{1 + \rho}} \sin \Delta i_1 \sqrt{1 + 2\rho - 2 \frac{2\rho}{1 + \rho} \cos \Delta i_1} = \frac{1}{\sqrt{\rho}} \sqrt{\frac{2\rho}{1 + \rho}} \sin (\Delta i - \Delta i_1) \sqrt{1 + \frac{2\rho}{1 + \rho} - 2 \frac{2\rho}{1 + \rho} \cos (\Delta i - \Delta i_1)}$$

$$\frac{2\rho}{1 + \rho} \sin^2 \Delta i_1 \left[1 + \frac{2}{1 + \rho} - 2 \frac{2\rho}{1 + \rho} \cos (\Delta i - \Delta i_1)\right] = \frac{1}{\rho} \frac{2\rho}{1 + \rho} \sin^2 (\Delta i - \Delta i_1) \left[1 + \frac{2\rho}{1 + \rho} - 2 \frac{2\rho}{1 + \rho} \cos \Delta i\right]$$

$$\rho = \frac{42200}{6370 + 500} = 6.14265 \quad \frac{2\rho}{1 + \rho} = 1.31148$$

$$1.31148 \sin \Delta i_1 \sqrt{1 + 1.71999 - 2 \times 1.31148 \cos \Delta i_1} = \frac{0.52916 \sin (28.5 - \Delta i_1)}{6.14265 \sqrt{1 + 0.28001 - 2 \times 0.52916 \cos (28.5 - \Delta i_1)}}$$
\[
\begin{align*}
\frac{\sin \Delta \iota_1}{\sqrt{2.71999 - 2.62296 \cos \Delta \iota_1}} &= \frac{0.16280 \sin(28.5 - \Delta \iota_1)}{\sqrt{1.28001 - 1.05832 \cos(28.5 - \Delta \iota_1)}} \\
\Delta \iota_1 &= 2.26^\circ \text{ optimum} \\
\Delta \iota_2 &= 26.24^\circ
\end{align*}
\]

\[
\left( \frac{\Delta V}{V_{c_1}} \right)_{op} = \sqrt{1 + \frac{2\rho}{1 + \rho} - 2\frac{2\rho}{1 + \rho} \cos \Delta \iota_1} + \frac{1}{\sqrt{\rho}} \left( \frac{1 + 2\rho}{1 + \rho} - \frac{2}{\sqrt{6.14265}} \right) \cos \Delta \iota_2
\]

\[
= 0.30178 + 0.23227 = 0.53405 \quad \text{- small improvement}
\]

Compare to same with \( \Delta \iota_1 = 0 \)

\[
\left( \frac{\Delta V}{V_{c_1}} \right)_{ref} = 0.29838 + 0.23868 = 0.53706 \quad \text{- small improvement}
\]
Example: Effects of doing a small plane change \( \Delta i_2 \) simultaneous with the second (apogee-raising) firing in a 3-impulse direct GTO injection.

Total \( dV \) for three-impulse launch from \( L=28.5 \) deg to GEO. Here \( v_cE = \sqrt{\mu/RE} \)
dV1 for three-impulse launch from L=28.5 deg to GEO. Here \( v_{CE} = \sqrt{\mu/RE} \)
dV1 for three-impulse launch from L=28.5 deg to GEO. Here vCE=\sqrt{\mu/RE}
$dV_3/vc_E$ for three-impulse launch from $L=28.5$ deg. to GEO. Here, $vc_E=\sqrt{\mu/RE}$