The Electron Energy Equation

Since \( \bar{v}_e \) is often very different from other species mean velocities, it makes sense to formulate the energy equation in terms of \( T'_e \), defined as \( n_e \frac{3}{2} k T'_e = \left\langle \frac{1}{2} m_v c_e^2 \right\rangle_e, \) \( \bar{v}_e = \bar{w} - \bar{v}_e \). We found before (with no inelastic effects)

\[
\frac{\partial}{\partial t} \left( n_e \frac{3}{2} k T'_e \right) + \nabla \cdot \left( n_e \bar{v}_e \frac{3}{2} k T'_e + \bar{q}_e \right) + \frac{\mathbf{P}}{\rho} \cdot \nabla \bar{v}_e = \sum_r E'_re
\]  

(1)

and, at least for Maxwellian collision (but generalizable to others),

\[
E'_re = n_e v_{re} \mu_{re} \left[ \frac{3k(T'_r - T'_e)}{m_e + m_r} + \frac{m_r}{m_r + m_e} (\bar{v}_r - \bar{v}_e)^2 \right] \]  

(2a)

\[
E'_re \approx n_e v_{re} m_e \left[ \frac{3k(T'_r - T'_e)}{m_r} + (\bar{v}_r - \bar{v}_e)^2 \right] \]  

(2b)

where (2b) results from \( m_e << m_r \).

One first observation is that the “heat transfer” portion of this, namely

\[
E_l = n_e v_{re} \frac{2m_e}{m_r} \frac{3}{2} k(T'_r - T'_e)
\]  

(3)

can be thought of as transferring the mean thermal energy difference per particle, \( \frac{3}{2} k(T'_r - T'_e) \), per collision, but with a very poor efficiency

\[
n_{el} = \frac{2m_e}{m_r} << 1
\]  

(4)

In other words, while Maxwellian distributions about \( T'_e \) and \( T'_r \) established in a few e.g. e-e and r-r collisions, it takes about \( \frac{m_r}{2m_e} \) collisions (tens to hundreds of thousands) to drive \( T'_e \)
toward $T'_r$. In practice, a good approximation is that there are separate Maxwellian populations for electrons (at $T'_e$) vs. the heavy species (at close to the same $T'_r$, since the collisional efficiency among them $\frac{2\mu_{re}}{m_r + m_s}$ is the of order 1).

We next examine the irreversible second term in (2b). The summation over r-species includes ions and neutrals, and we assume a single kind of each. We then have a dissipation

$$D = n_e m_e [v_{ei}(\vec{v}_i - \vec{v}_e)^2 + v_{en}(\vec{v}_n - \vec{v}_e)^2]$$  \hspace{1cm} (5)$$

The electron and ion current densities are

$$j_e = -en_e \vec{v}_e$$  \hspace{1cm} (6a)$$

$$j_i = en_e \vec{v}_i$$  \hspace{1cm} (6b)$$

and so $$D = n_e m_e \left[ v_{ei} \left( \frac{j_e}{en_e} \right)^2 + v_{en} \left( \vec{v}_n + \frac{j_e}{en_e} \right)^2 \right]$$  \hspace{1cm} (7)$$

This expression simplifies several limits:

(a) $v_n << v_i, v_e$

$$D \approx n_e m_e \left[ v_{ei} \frac{j^2}{e^2 n^2_e} + v_{en} \frac{j^2}{e^2 n^2_e} \right] = \frac{j^2}{\left( \frac{e^2 n_e}{m_e v_{ei}} \right) + \left( \frac{e^2 n_e}{m_e v_{en}} \right)}$$

The quantities in the denominator are the conductivities if only ei or en collisions occurred:

$$D \approx \frac{j^2}{\sigma_{ei}} + \frac{j^2}{\sigma_{en}}$$  \hspace{1cm} (8)$$

and, in particular

(a.1) For a neutral-dominated gas ($v_{en} >> v_{ei}$) (Hall Thruster), $D \approx \frac{j^2}{\sigma}$  \hspace{1cm} (9)$$

(a.2) For a Coulomb-dominated gas ($v_{ei} >> v_{en}$) $D \approx \frac{j^2}{\sigma}$  \hspace{1cm} (10),
(a.3) If $j_i << j_e, j_e \simeq j$ and $D \simeq \frac{j^2}{\sigma}$, with $\sigma = \frac{e^2n_e}{m_e(v_{ei} + v_{en})}$.

(b) When density is relatively high, ions and neutrals couple strongly and $\vec{v}_n \simeq \vec{v}_i$ (as in MPD thrusters or MHD generators). In that case (5) yields

$$D \simeq n_em_e(v_{ei} + v_{en})(\vec{v}_i - \vec{v}_e)^2 = \frac{j^2}{\sigma}$$

with

$$\sigma = \frac{e^2n_e}{m_e(v_{ei} + v_{en})}$$

One approximation which is routinely made is to neglect the viscous dissipation of the electron gas, i.e., the contribution of the off-diagonal terms in $P^e \cdot \nabla \vec{v}_e$:

$$\Rightarrow P^e \cdot \nabla \vec{v}_e \simeq P^e_v \nabla \cdot \vec{v}_e$$

where $P^e_v = n_e kT_e'$ is the scalar pressure (the trace of $\Rightarrow$). Breaking this into $\nabla \cdot (n_e kT_e' \vec{v}_e) - \vec{v}_e \cdot \nabla P^e_v$ and substituting in (1), we get

$$\frac{\partial}{\partial t} \left( n_e \frac{3}{2} kT_e' \right) + \nabla \cdot \left( n_e v_e \frac{5}{2} kT_e' + \vec{q}_e' \right) = D + El + \vec{v}_e \cdot P^e_v$$

with $D$ given by (7) and $El$ given by (3).

Finally, although we will not prove it here, it stands to reason that the heat flux vector $\vec{q}_e = n_e \left( \frac{1}{2} m_e c^2 c_e \right) / e$ will be expressible in the form of a Fourier law

$$\vec{q}_e = -K_e(T_e) \nabla T_e'$$

where $K_e$ is the electron thermal conductivity. Note that a simple form like this may not be accurate when the alternative definition $T_e$ of temperature is used.