Probabilistic Analysis and Optimization

• In this last module we will cover:
  – Probabilistic analysis via Monte Carlo simulation
  – Monte Carlo convergence and error estimation
  – Statistical sampling
  – Design of experiments
  – Sensitivity analysis
  – Introduction to design optimization
Measurable Outcomes

- Estimate gradient with finite difference
  - Interpret sensitivity information
  - Describe methods to estimate gradients

- Sample arbitrary univariate distribution
  - Use Monte Carlo sampling
    - Define bias/variance of an estimator
    - Be an estimator
    - Obtain confidence intervals
    - State unbiased estimators
    - Define standard error / sampling distrib
    - Standard errors for sample estimators

- Describe the $R^2$-metric
  - Describe the response surface method
    - Construct a response surface

- Describe Newton's method: describe the conjugate gradient method
  - Apply optimization techniques
    - Describe the steepest descent method
Today’s Topics

1. Importance of probabilistic analysis in aerospace design
2. Monte Carlo (MC) methods
3. Probability & statistics refresher
4. Turbine blade heat transfer example
5. MC method for uniform distributions
6. MC method for non-uniform distributions
Characterizing, representing and analyzing uncertainty in simulation tools is essential for aerospace systems

- To support decision-making (optimization, control, design, policy)
- To inform model development
2. Monte Carlo methods

- A way of determining how uncertainty in inputs translates to uncertainty in outputs.

Nome: coined by researchers at Los Alamos as a joke: Canio where one's uncle used to gamble.

Most famous early use: Enrico Fermi 1930 use to compute properties of newly discovered neutron.

General steps in MC methods:
1. Define distributions of inputs
2. Sample inputs randomly, run deterministic solve on each
3. Analyze resulting distributions of outputs to estimate desired statistical outputs

(e.g. BCS, ICS, forcing terms)

(system)

(inputs) → (outputs)

(temperature, velocity, lift, drag etc.)
3. Probability & statistics refresher

**PDF:** If $X$ is a random variable, then

$$P \{ a \leq X \leq b \} = \int_a^b f(x) \, dx$$

- Probability that $X$ lies between $a$ and $b$
- **PDF of $X$**

**Example:** Uniform

- $f(x) = \frac{1}{x_u-x_l}$ for $x \in [x_l, x_u]$
- $0$, otherwise

**Example:** Normal

- $f(x) = \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$
- mean $\mu$, s.d. $\sigma$
- $X \sim N(\mu, \sigma)$
4. Turbine blade heat transfer example

\[ T_{\text{gas}} \]  
\[ k_{\text{BC}} \]  
\[ L_{\text{BC}} \]  
\[ T_{\text{mh}} \]  
\[ T_{\text{cool}} \]

Inputs: \( h_{\text{gas}}, T_{\text{gas}}, k_{\text{BC}}, L_{\text{BC}}, k_{m}, L_{m}, h_{\text{cool}}, T_{\text{cool}} \).

Eqs. (5-1-4) solve to determine \( T_{\text{BC}}, T_{\text{mh}}, T_{\text{cool}} \).

Output of interest: \( T_{\text{mh}} \), hot-side metal temp.

Deterministic calculation: Ex. 15-1 \( \Rightarrow T_{\text{mh}} = 1121 - 8 \text{ K} \)

We will consider impact of variability on \( L_{\text{BC}} \) on \( T_{\text{mh}} \).
Turbine blade heat transfer example

In our MC approach we will:
1. Define distribution for \( L_{TBC} \) (knowledge of manufacturing process)
2. Sample randomly from our \( L_{TBC} \) distribution, analyze (deterministically) each case in the sample
3. Collect the value of \( T_{mn} \) for each case run. Use these \( T_{mn} \) data to estimate e.g.
   \( \rightarrow \) distribution of \( T_{mn} \) would be observed in population of infused blades
   \( \rightarrow \) prop. that \( T_{mn} \) exceeds some critical value
   \( \rightarrow \) mean value of \( T_{mn} \), \( \mu_{Tmn} \)
   \( \rightarrow \) std. dev. of \( T_{mn} \), \( \sigma_{Tmn} \)
5. MC method for uniform distributions

Step 1: Assume \( L_{TBC} \) is uniformly distributed from 
\[ 0.00025 \, \text{m} < L_{TBC} < 0.00075 \, \text{m} \]

\[ \text{PDF}(L_{TBC}) \]

\[ 0.25 \quad 0.75 \quad L_{TBC} \times 10^{-3} \]

Step 2: (a) Sample. To generate random numbers \( u \) in Matlab "rand"

\[ \text{returns} \quad u \quad \text{pdf}(u) \]

\[ \Rightarrow \text{then we can use} \quad L_{TBC} = 0.00025 + 0.0005u \]
Simulation Challenge

• You are given the model in the file “blade1D.m”
  – function \([Ttbc, Tmh, Tmc, q] = \text{blade1D}(hgas, Tgas, ktbc, Ltbc, km, Lm, hcool, Tcool)\)

• Write a Monte Carlo simulator

• Run these cases with \(N=10, 100, 1000:\)
  – \(LTBC \sim \text{U}(0.00025, 0.00075)\)
  – Other variables deterministic at values given in HW8
    Hint: the Matlab function \(\text{rand}\) returns a uniformly distributed random number on the interval \((0,1)\).

• Generate an output histogram for \(Tmh\) and estimate the output mean

• Qualitatively, what evidence is there that your MCS implementation is correct?
6. MC method for non-uniform distributions

- How to generate random nos. w/ non-uniform dists?

We use inversion method.

CDF:

\[ F(x) = \int_{-\infty}^{x} f(x) \, dx \]

\[ F(x) = P[X \leq x] \]

Inversion method:

1. Generate \( u \) from \( U(0, 1) \)
2. Given \( u \), find value of \( x \) at which \( u = F(x) \)

i.e. \( x = F^{-1}(u) \)
Triangular distributions