Iterative Methods:
Multigrid Techniques
Lecture 7
Background

– Developed over the last 25 years — Brandt (1973) published first paper with practical results.
– Offers the possibility of solving a problem with work and storage proportional to the number of unknowns.
– Well developed for linear elliptic problems — application to other equations is still an active area of research.

Basic Principles

Some ideas

1. Multigrid is an iterative method → a good initial guess will reduce the number of iterations:

   to solve $A_h u_h = f_h$ by iteration, we could take $u_0^h \sim u_{2h}$, where $A_{2h} u_{2h} = f_{2h}$ …

   but … the number of iterations needed to solve $A_h u_h = f_h$ still $O(n^2)$.

   $h = \frac{1}{n+1}$
2. If after a few iterations, the error is smooth, we could solve for the error on a coarser mesh, e.g., $A_{2h} e_{2h} = r_{2h}$.

- Smooth functions can be represented on coarser grids;
- Coarse grid solutions are cheaper.
If the *high frequency* components of the error decay faster than the *low frequency* components, we say that the iterative method is a *smoother*. 
Basic Principles

Is Jacobi a smoother?  \(\text{...} \rightarrow \text{NO}\)

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Basic Principles

\[ R_{\omega J} = \omega R_J + (1 - \omega) I \]

\[ \lambda^k(R_{\omega J}) = \omega \lambda^k(R_J) + (1 - \omega) = 1 - \omega (1 - \lambda^k(R_J)), \]

\[ k = 1, \ldots, n \]
Basic Principles

Smoothing
...Under-Relaxed Jacobi

Iterations required to reduce an error mode by a factor of 100
Basic Principles

Recall,

Is Gauss-Seidel a good smoother?
Basic Principles

Smoothers

...Gauss-Seidel

Iterations required to reduce an *A error mode* by a factor of 100

...GS is a *good smoother*.
Given $w_h$ we obtain $w_{2h}$ by restriction

$$w_{2h} = I_{2h}^h w_h$$

$I_{2h}^h$: restriction operator (matrix).

Simplest procedure is injection

$$w_{2h,i} = w_{h,2i} \quad \text{for} \quad i = 1, \ldots, \frac{n-1}{2}$$
Intuitively,
If we write

\[ w_h = \sum_{k=1}^{n} c_k v^k \]

Then only the modes \( k = 1, \ldots, \frac{n-1}{2} \) are "visible" by grid \( 2h \).

“visible” by grid \( 2h \):

\[ 1, 2, \ldots, \frac{n-1}{2}, \quad \frac{n+1}{2}, \ldots, n-1, n \]

aliased
Mode $k > (n - 1)/2$ on grid $h$ becomes $(n - k)$ mode on grid $2h$. 
Basic Principles

- Only low modes in \( h \) can be represented well in \( 2h \).
- Low modes on \( h \) become higher modes in \( 2h \).

<table>
<thead>
<tr>
<th>( k = 1 )</th>
<th>( \frac{n-1}{2} )</th>
<th>( \frac{n+1}{2} )</th>
<th>( n )</th>
</tr>
</thead>
<tbody>
<tr>
<td>LOW</td>
<td>HIGH</td>
<td>grid ( h )</td>
<td></td>
</tr>
<tr>
<td>LOW</td>
<td>HIGH</td>
<td></td>
<td></td>
</tr>
<tr>
<td>LOW</td>
<td>HIGH</td>
<td>( 2h )</td>
<td></td>
</tr>
<tr>
<td>LOW</td>
<td>HIGH</td>
<td>( 4h )</td>
<td></td>
</tr>
</tbody>
</table>

\[ \vdots \]
Given $w_{2h}$ we obtain $w_h$ by prolongation

$$w_h = I_h^{2h} w_{2h}$$

$I_h^{2h}$: prolongation operator (matrix).

Typically, we use interpolation.

$$i = 1, \ldots, \frac{n-1}{2}$$

$$w_{h,2i} = w_{2h,i}$$

$$w_{h,2i+1} = \frac{1}{2} (w_{2h,i} + w_{2h,i+1})$$
Basic Principles

Prolongation

\[ 2h \]

\[ h \]
Basic Principles

Interpolation introduces high frequency errors.
Two Grid (Correction) Scheme

One cycle

\[ u_{h}^{r+1} \leftarrow MG(u_{h}^{r}, f_{h}) \]

- Relax \( \nu_1 \) iterations of \( A_{h} u_{h} = f_{h} \) with initial guess \( u_{h}^{r} \rightarrow u_{h}^{r+1/3} \).

- Compute \( r_{h} = f - A_{h} u_{h}^{r+1/3} \), and restrict \( r_{2h} = I_{2h}^{h} r_{h} \).

- Solve \( A_{2h} e_{2h} = r_{2h} \) on \( 2h \).

- Prolongate \( e_{h} = I_{h}^{2h} e_{2h} \), and correct \( u_{h}^{r+2/3} = u_{h}^{r+1/3} + e_{h} \).

- Relax \( \nu_2 \) iterations of \( A_{h} u_{h} = f_{h} \) with initial guess \( u_{h}^{r+2/3} \rightarrow u_{h}^{r+1} \).
We solve \( u(0) = u(1) = 0 \)
\[-u_{xx} = -25(\sin(5\pi x) + 9 \sin(15\pi x)) .\]

Initial guess: \( u^0 = 0 \)

Solution: \( u = \sin(5\pi x) + \sin(15\pi x) \)

Two grid scheme: \( h = \frac{1}{32}, \quad 2h = \frac{1}{16} \)

Solve using under-relaxed Jacobi with \( \omega = \frac{2}{3} \)
Two Grid (Correction) Scheme

Initial condition

Example
Two Grid (Correction) Scheme

After $\nu_1 = 2$ iterations on the fine mesh
After coarse grid correction (4 iterations)
Two Grid (Correction) Scheme

After $\nu_2 = 2$ post smoothing iterations (end of cycle 1)
Two Grid (Correction) Scheme

After $\nu_1 = 2$ iterations
Two Grid (Correction) Scheme

After coarse grid correction
Two Grid (Correction) Scheme

After $\nu_2 = 2$ iterations (end of cycle 2)
Two Grid (Correction) Scheme

Multigrid convergence vs. single grid
Multiple Grids

One cycle

- Relax $\nu_1$ times on $A_h u_h = f_h$ with initial guess $u_h^r \rightarrow u_h^{r+1/3}$.
- If $h \equiv$ coarsest grid, go to (SKIP)
  Else
    \[ r_{2h} \leftarrow I_{2h}^h (f_h - A_h u_h^{r+1/3}) \]
    \[ e_{2h} \leftarrow V G_{2h}(0, r_{2h}) . \]
- Correct $u_h^{r+2/3} = u_h^{r+1/3} + I_{h}^{2h} e_{2h}.$
- (SKIP) Relax $\nu_2$ times on $A_h u_h = f_h$ with initial guess $u_h^{r+2/3} \rightarrow u_h^{r+1}.$
Multiple Grids

V-Cycle Schematically

Diagram showing a V-shaped grid with labels at each level: $h$, $2h$, $4h$, and $8h$. Each level represents a grid with a coarser resolution.
Solve
\[-(u_{xx} + u_{yy}) = 1, \quad \in \Omega \equiv \text{unit square}\]

\[u = 0 \quad \text{on the boundary}\]
Multiple Grids

V-Cycle

...2D Example...

Parameter dependence
Multiple Grids

V-Cycle

...2D Example...

Convergence as a function of grid levels (same fine mesh)
Multiple Grids

V-Cycle

...2D Example

Convergence as a function of grid levels (same coarse mesh)
Multiple Grids

$W$-Cycles
Full Multigrid Scheme

Schematically

Putting it all together . . .
More Advanced Topics

- Anisotropic grids/equations.
- Algebraic multigrid.
- Convergence theory.
- How to deal with other operators.