Problem Set 7
1. Consider the following uncertain nonlinear dynamical model of a vehicle:

\[
\begin{bmatrix}
  x \\
  y \\
  \psi
\end{bmatrix}, \quad \dot{x} = f(x, u, w) = \begin{bmatrix}
  \dot{x} \\
  \dot{y} \\
  \dot{\psi}
\end{bmatrix} = \begin{bmatrix}
  -v \sin \psi + w \\
  v \cos \psi \\
  u
\end{bmatrix}
\]

- Control inputs \( u, v \)
- Source of Uncertainty: disturbance \( \omega \in [-0.1, 0.1] \)
- Obstacle: \( X_{obs} = \{ (x_1, x_2): 0.25^2 - x_1^2 - (x_2 - 0.5)^2 \geq 0 \} \)

- Motion Primitives:
  1) \( \psi_1^* = 0, \quad v_1 = 1m/s, u_1 = -50(\psi - \psi_1^*), \quad \psi_2^* = \frac{15\pi}{180}, \quad v_2 = 1.5m/s, u_2 = -50(\psi - \psi_2^*) \)
  2) \( \psi_3^* = \frac{45\pi}{180}, \quad v_1 = 2m/s, u_3 = -50(\psi - \psi_3^*), \quad \psi_4^* = \frac{90\pi}{180}, \quad v_4 = 3m/s, u_4 = -50(\psi - \psi_4^*) \)

Check the safety of the given motion primitives.

**Hint:** Similar example on the page 33 of Lecture 8.
2. Consider the following uncertain nonlinear dynamical system

\[
\begin{align*}
    x_1(k+1) &= x_2(k) \\
    x_2(k+1) &= x_1(k)x_2(k) + u(k) + (0.2\omega(k) - 0.1)
\end{align*}
\]

- Source of uncertainties at time k: \((x_1(k), x_2(k), \omega(k)) \in \Omega_x = \{(x_1, x_2, \omega) = 0.1^2 - x_1^2 - x_2^2 - \omega^2 \geq 0\}.

- Goal Set: Neighborhood of the way-point (0,0.5), i.e. a ball around the way-point

\[
X_{safe} = \{ (x_1, x_2): 0.2^2 - (x_1 - 0)^2 - (x_2 - 0.5)^2 \geq 0 \}
\]

- Robust set for control input at time \(k\):

\[
U_R = \{ u(k) : x(k+1) \in X_{safe}, \forall \omega \in \Omega_x \}
\]

i) Find the inner approximation of the robust set of control input \(U_R\) using the SOS program with relaxation order \(d = 4\).

ii) Obtain the Robust Set using Monte-Carlo Approach.

**Hint:** Similar example on the page 64 of Lecture 8.
3. Consider the following uncertain nonlinear dynamical system

\[
\begin{align*}
    x_1(k + 1) &= x_2(k) \\
    x_2(k + 1) &= x_1(k)x_2(k) + u(k) + (0.2\omega(k) - 0.1)
\end{align*}
\]

Source of uncertainties at time \( k \):

\[
(x_1(k), x_2(k)) \sim U([-0.1,0.1]^2) \quad \omega_k \sim N(m,\sigma), \ m \in [-0.1,0.1], \sigma \in [0.1,0.3]
\]

Unsafe set:

\[
X_{obs} = \{ (x_1, x_2): 0.3^2 - (x_1 - 0.2)^2 - (x_2 - 0.3)^2 \geq 0 \}
\]

Distributionally Robust Chance constrained set for control input at time \( k \):

\[
U_{DR} = \{ u(k) : \text{Prob}(x(k + 1) \in X_{safe}) \geq 1 - \Delta, \ \forall N(m,\sigma) \}
\]

\( \Delta = 0.2 \)

i) Find the inner approximation of the set \( U_{DR} \) using the SOS program with relaxation order \( d = 6 \).

ii) Choose a control input \( u_k \in U_{DR} \). Using the Monte-Carlo approach show that for the chosen controller the following result holds true:

\[
\text{Prob}(x(k + 1) \in X_{safe}) \geq 1 - \Delta
\]

**Hint:** Similar example on the page 121 of Lecture 8.