### Table 2.7 Complete Description of Electrodynamics

<table>
<thead>
<tr>
<th>Name</th>
<th>Differential form</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1) Gauss’ law</td>
<td>$\nabla \cdot \varepsilon \mathbf{E} = \rho_e$</td>
</tr>
<tr>
<td>(2) Faraday’s law</td>
<td>$\nabla \times \mathbf{E} = -\frac{\partial \mu \mathbf{H}}{\partial t}$</td>
</tr>
<tr>
<td>(3) Ampère’s law</td>
<td>$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \varepsilon \mathbf{E}}{\partial t}$</td>
</tr>
<tr>
<td>(4) Magnetic flux</td>
<td>$\nabla \cdot \mu \mathbf{H} = 0$</td>
</tr>
<tr>
<td>(5) Charge conservation</td>
<td>$\nabla \cdot \mathbf{J} = -\frac{\partial \rho_e}{\partial t}$</td>
</tr>
<tr>
<td>(6) Lorentz force law</td>
<td>$\mathbf{F} = \rho_e \left( \mathbf{E} + \mathbf{v} \times \mu \mathbf{H} \right)$</td>
</tr>
<tr>
<td>(7) Newton’s law (single charged particle)</td>
<td>$m \left( \frac{\partial \mathbf{v}}{\partial t} \right) = q \left( \mathbf{E} + \mathbf{v} \times \mu \mathbf{H} \right) + \mathbf{f} \text{ other}$</td>
</tr>
</tbody>
</table>

**Constitutive Laws for Linear, Isotropic Media**

- $\mathbf{D} = \varepsilon \mathbf{E} = \varepsilon_0 \mathbf{E} + \mathbf{P}$
- $\mathbf{B} = \mu \mathbf{H}$
- $\mathbf{J} = \sigma \mathbf{E}$
Table 2.7 Complete Description of Electrodynamics

<table>
<thead>
<tr>
<th>Name</th>
<th>Differential form</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1) Gauss’ law</td>
<td>$\nabla \cdot \varepsilon \mathbf{E} = \rho_e$</td>
</tr>
<tr>
<td>(2) Faraday’s law</td>
<td>$\nabla \times \mathbf{E} = -\frac{\partial \mu \mathbf{H}}{\partial t}$</td>
</tr>
<tr>
<td>(3) Ampere’s law</td>
<td>$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \varepsilon \mathbf{E}}{\partial t}$</td>
</tr>
<tr>
<td>(4) Magnetic flux</td>
<td>$\nabla \cdot \mu \mathbf{H} = 0$</td>
</tr>
<tr>
<td>(5) Charge conservation</td>
<td>$\nabla \cdot \mathbf{J} = -\frac{\partial \rho_e}{\partial t}$</td>
</tr>
<tr>
<td>(6) Lorentz force law</td>
<td>$\mathbf{F} = \rho_e \left( \mathbf{E} + \mathbf{v} \times \mu \mathbf{H} \right)$</td>
</tr>
<tr>
<td>(7) Newton’s law: single charged particle</td>
<td>$m \left( \frac{\partial \mathbf{v}}{\partial t} \right) = q(\mathbf{E} + \mathbf{v} \times \mu \mathbf{H}) + \mathbf{f}_{\text{other}}$</td>
</tr>
</tbody>
</table>

Constitutive Law

$\mathbf{J} = \sigma \mathbf{E} + \ldots$

either $\mathbf{H} = 0$, or
- $\partial / \partial t \approx$ small
- low enough freq
- $\lambda >> L_{\text{char}}$

© Garland Science. All rights reserved. This content is excluded from our Creative Commons license. For more information, see [http://ocw.mit.edu/help/faq-fair-use/](http://ocw.mit.edu/help/faq-fair-use/).
Ohm's Law:

\[ J = \sigma E \]

( empirical )

1789-1854

Mathematician and experimentalist

Current Flow in conductors
Electro-Statics:

0.1 M NaCl
pH 7 (initially)

platinum electrodes

x = 0  x = L
y = 0  y = d

$V_o$
But: Electrolysis Reactions at Electrodes

Really: \[ J = \sigma E + \text{diffusion} + \text{convection} \]
**ElectroStatics:** $\nabla \cdot J = - (\partial \rho_e / \partial t) \equiv 0$

$\nabla \cdot J = 0 = \nabla \cdot \sigma E = \sigma [\nabla \cdot (-\nabla \Phi)] = 0 \rightarrow \nabla^2 \Phi = 0 \text{ Laplace}$

$\Phi = V_o (1 - x/L)$

$E = -\nabla \Phi = \hat{i}_x (V_o/L)$

$x = 0 \rightarrow V=0$

$\text{platinum electrodes}$

$\text{(conductivity, } \sigma) \quad \text{(permittivity, } \varepsilon)$
Table B.7

Solutions of Laplace’s Eq.

\( \nabla^2 \Phi = 0 \) in 2-dimen’s

**Rectangular Coordinates** (independent of \( z \))

\( e^{kx} \) and \( e^{-kx} \) may be replaced by \( \sinh kx \) and \( \cosh kx \).

\[
\Phi = e^{kx}(A_1 \sin ky + A_2 \cos ky) + e^{-kx}(B_1 \sin ky + B_2 \cos ky)
\]

\[
\Phi = Axy + Bx + Cy + D; \quad (k = 0)
\]

**Cylindrical Coordinates** (independent of \( z \))

\[
\Phi = r^n(A_1 \sin n\phi + A_2 \cos n\phi) + r^{-n}(B_1 \sin n\phi + B_2 \cos n\phi)
\]

\[
\Phi = (A_1 \phi + A_2) \ln \frac{R}{r} + B_1 \phi + B_2; \quad (n = 0)
\]

**Spherical Coordinates** (independent of \( \phi \)):

\[
\Phi = A r \cos \theta + \frac{B}{r^2} \cos \theta + \frac{C}{r} + D
\]
PSet 4, P3: Gradient Gel Electrophoresis

∇^2 \Phi = 0

Splice solutions together via appropriate Boundary Conditions

\Phi = 0

\Phi = + V_0

y = d

y = 0

x = 0

X = L/2

X = L

Plastic: conductivity \sigma = 0

\sigma, \varepsilon

\sigma/2, \varepsilon

(J = \sigma E)

(J = \sigma E)
LAW

(1) \nabla \cdot \varepsilon \mathbf{E} (r, t) = \rho_e (r, t)

Gauss

(2) \nabla \times \mathbf{E} = 0

Faraday

\Rightarrow \mathbf{E} = -\nabla \Phi

(3) \nabla \cdot \mathbf{J} (r, t) = -\frac{\partial \rho_e (r, t)}{\partial t}

Cons. of Charge

(4) \mathbf{J} = \sigma \mathbf{E} (+ \text{other}) \text{ (constitutive law)}

Boundary Cond.

(1') \mathbf{n} \cdot (\varepsilon_1 \mathbf{E}_1 - \varepsilon_2 \mathbf{E}_2) = \sigma_s (t)

(2') \mathbf{n} \times (\mathbf{E}_1, -\mathbf{E}_2) = 0

\Rightarrow \Phi_1 = \Phi_2

(E_{\text{tan cont}} \text{inuous})

(3') \mathbf{n} \cdot (\mathbf{J}_1 - \mathbf{J}_2) = -\frac{\partial \Phi (t)}{\partial t}

\Rightarrow \mathbf{n} \cdot (\sigma_1 \mathbf{E}_1 - \sigma_2 \mathbf{E}_2) = -\frac{\partial \Phi (t)}{\partial t}

0 \text{ static}
Electro-Statics:

\[ \nabla \cdot \varepsilon \mathbf{E} = \rho_e = 0 \quad \rightarrow \quad \nabla^2 \Phi = 0 \quad \text{Laplace} \]

\[ \rho_e \text{ (coul)} = \sum_i \frac{z_i F \text{ (coul)} c_i \text{ (mol)}}{m^3} \text{ mol} \frac{m^3}{m^3} \]

\[ \rho_e = 0 \text{ in “bulk”} \]

\[ x = 0 \quad x = L \]

0.1 M NaCl
pH 7 (initially)
But is \( \rho_e \) zero everywhere?

\[
\nabla \cdot \varepsilon \mathbf{E} = \rho_e = 0 \rightarrow \nabla^2 \Phi = 0 \quad \text{Laplace}
\]

\( \rho_e = 0 \) in “bulk”

\( 0.1 \text{ M NaCl} \)

pH 7 (initially)

\( J = \sigma \mathbf{E} \) (in bulk)

x = 0

x = L

(platinum electrodes)
Poisson-Boltzmann: Molecular Electrostatic Interactions

\[
\text{COO}^- + \text{H}^+ \leftrightarrow \text{COOH}
\]

Intra-molecular Electrostatic Interactions

Inter-molecular Electrostatic Interactions
In Figure 2.24, we picture an idealized metal electrode/electrolyte interface where the metal is known to have a net surface charge $\sigma_d$ at $x = 0$. This leads to a net space charge of mobile ions in the adjacent electrolyte phase. We wish to find the *equilibrium* potential and space charge distribution in the electrolyte.

(a) For the one-dimensional model of Figure 2.24, write Poisson’s equation for the electrolyte region $x \geq 0$.

Further, assume that the distribution of all mobile ions can be adequately represented by Boltzmann statistics, so that the probability of finding a given ion of species $i$ and valence $z_i$ at position $x$ can be written as $\exp[-z_iF\Phi(x)/RT]$, and therefore

\[
\Phi(x) \to 0
\]
**Table 2.7 Complete Description of Electrodynamics**

<table>
<thead>
<tr>
<th>Name</th>
<th>Differential form</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1) Gauss’ law</td>
<td>$\nabla \cdot \varepsilon E = \rho_e$</td>
</tr>
<tr>
<td>(2) Faraday’s law</td>
<td>$\nabla \times E = -\frac{\partial \mu H}{\partial t}$</td>
</tr>
<tr>
<td>(3) Ampere’s law</td>
<td>$\nabla \times H = J + \frac{\partial \varepsilon E}{\partial t}$</td>
</tr>
<tr>
<td>(4) Magnetic flux</td>
<td>$\nabla \cdot \mu H = 0$</td>
</tr>
<tr>
<td>(5) Charge conservation</td>
<td>$\nabla \cdot J = -\frac{\partial \rho_e}{\partial t}$</td>
</tr>
<tr>
<td>(6) Lorentz force law</td>
<td>$F = \rho_e \left[ E + \mathbf{v} \times \mu H \right]$</td>
</tr>
<tr>
<td>(7) Newton’s law:</td>
<td>$m \left( \frac{\partial \mathbf{v}}{\partial t} \right) = q(\mathbf{E} + \mathbf{v} \times \mu \mathbf{H}) + \mathbf{f}^{\text{other}}$</td>
</tr>
</tbody>
</table>

(single charged particle)

**Constitutive Law**

$\mathbf{J} = \sigma \mathbf{E} + \ldots$

$\approx 0$

either $\mathbf{H} = 0$, or
- $\partial / \partial t \approx$ small
- low enough freq
- $\lambda >> L_{\text{char}}$

---

© Garland Science. All rights reserved. This content is excluded from our Creative Commons license. For more information, see [http://ocw.mit.edu/help/faq-fair-use/](http://ocw.mit.edu/help/faq-fair-use/).

FFF: Complete Description of Coupled Transport and Biomolecular Interactions

\[ \dot{N}_i = -D_i \nabla c_i + \frac{z_i}{z_j} u_i c_i \frac{E}{E} \]

Current Density \( J \)

Diffusion-Reaction

\[ \frac{\partial c_i}{\partial t} = -\nabla \cdot \vec{N}_i + R_{wi} \]

\[ \nabla \cdot \vec{E} = \rho_e = \sum z_i F_{c_i} \]

\[ (\vec{E} = -\nabla \Phi) \]

\[ \nabla \cdot \vec{J} = -\frac{\partial \rho_e}{\partial t} \]

\[ \vec{J} = \sum z_i F \vec{N}_i \]

"E.Q.S."