Introduction to Neural Computation

Prof. Michale Fee
MIT BCS 9.40 — 2018

Lecture 11 - Spectral analysis I
Spectral Analysis

Frequency (Hz)

Time (s)
Game plan for Lectures 11, 12, and 13 —

*Develop a powerful set of methods for understanding the temporal structure of signals*

- Fourier series, Complex Fourier series, Fourier transform, Discrete Fourier transform (DFT), Power Spectrum
- Convolution Theorem
- Noise and Filtering
- Shannon-Nyquist Sampling Theorem
- Spectral Estimation
- Spectrograms
- Windowing, Tapers, and Time-Bandwidth Product
- Advanced Filtering Methods
Learning Objectives for Lecture 11

- Fourier series for symmetric and asymmetric functions
- Complex Fourier series
- Fourier transform
- Discrete Fourier transform (Fast Fourier Transform - FFT)
- Power spectrum
Discrete Fourier transform

• Some code

```matlab
N=2048;  % number of samples in time

dt=.001;  % sampling interval
Fs=1./dt;  % sampling frequency
time=dt*[-N/2:N/2-1];  % timebase

freq=20.;  % frequency of sine wave in Hz
y=cos(2*pi*freq*time);

yshft=circshift(y,[0,N/2]);  % First shift zero point from center to
fft=fft(yshft,N)/N;  % first point in the array
% Now compute the FFT

Y=circshift(fft,[0,N/2]);  % Now shift the spectrum to put zero frequency
% at the middle of the array

% Compute the vector of frequencies
df=Fs/N;
Fvec=df*[-N/2:N/2-1];
```
Fourier Series

• We can express any periodic function of time as sums of sine and cosine functions.

• Let’s start with an even function that is periodic with a period $T$

We could approximate this square wave with a cosine wave of the same period $T$ and amplitude.

$$a_1 \cos(2\pi f_0 t)$$

Oscillation frequency

$$f_0 = \frac{1}{T}$$

Cycles per second (Hz)

Angular frequency

$$\omega_0 = \frac{2\pi}{T}$$

Radians per second
Fourier Series

- But we can get a better approximation if we add some more cosine waves to our original one...

\[ y(t) = a_1 \cos(\omega_0 t) + a_2 \cos(2\omega_0 t) + a_3 \cos(3\omega_0 t) + ... \]
Fourier Series

\[ \cos(\omega_0 t) \]
\[ \cos(3\omega_0 t) \]
\[ \cos(5\omega_0 t) \]
\[ \cos(7\omega_0 t) \]
\[ \cos(9\omega_0 t) \]
\[ \cos(11\omega_0 t) \]
\[ \cos(13\omega_0 t) \]

-3 -2 -1 0 1 2 3

-3 -2 -1 0 1 2 3
Fourier Series

\[ \cos(n \omega_0 t) \]

\( n = \)

constructive interference
destructive interference

destructive interference

constructive interference
Fourier Series

\[ y(t) = \frac{a_0}{2} + a_1 \cos(\omega_0 t) + a_2 \cos(2\omega_0 t) + a_3 \cos(3\omega_0 t) + \ldots \]

\[ y_{even}(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\omega_0 t) \]
How do we find the coefficients?

- The $a_0/2$ coefficient is just like the average of our function $y(t)$.
  
  \[
  \frac{a_0}{2} = \frac{1}{T} \int_{-T/2}^{T/2} y(t) \, dt \\
  a_0 = \frac{2}{T} \int_{-T/2}^{T/2} y(t) \cos(0\omega_0 t) \, dt
  \]

- The $a_1$ coefficient is just the overlap of our function $y(t)$ with $\cos(\omega_0 t)$.
  
  \[
  a_1 = \frac{2}{T} \int_{-T/2}^{T/2} y(t) \cos(\omega_0 t) \, dt
  \]

- The $a_2$ coefficient is just the overlap of our function $y(t)$ with $\cos(2\omega_0 t)$.
  
  \[
  a_2 = \frac{2}{T} \int_{-T/2}^{T/2} y(t) \cos(2\omega_0 t) \, dt
  \]

- The $a_n$ coefficient is just the overlap of our function $y(t)$ with $\cos(n\omega_0 t)$.
  
  \[
  a_n = \frac{2}{T} \int_{-T/2}^{T/2} y(t) \cos(n\omega_0 t) \, dt
  \]
How do we find the coefficients?

\[
a_0 = \frac{2}{T} \int_{-T/2}^{T/2} y(t) \, dt \quad a_1 = \frac{2}{T} \int_{-T/2}^{T/2} y(t) \cos(\omega_0 t) \, dt \quad a_2 = \frac{2}{T} \int_{-T/2}^{T/2} y(t) \cos(2\omega_0 t) \, dt
\]

Consider the following functions \( y(t) \):

\[
y(t) = 1 \quad a_0 = 2 \quad a_1 = 0 \quad a_2 = 0
\]

\[
y(t) = \cos(\omega_0 t) \quad a_0 = 0 \quad a_1 = 1 \quad a_2 = 0
\]

\[
y(t) = \cos(2\omega_0 t) \quad a_0 = 0 \quad a_1 = 0 \quad a_2 = 1
\]

\[
\int_{-T/2}^{T/2} \left[ \cos(\omega_0 t) \right]^2 \, dt = \frac{T}{2} \quad \int_{-T/2}^{T/2} \cos(\omega_0 t) \cos(2\omega_0 t) \, dt = 0
\]

\[
y(t) = \frac{a_0}{2} + a_1 \cos(\omega_0 t) + a_2 \cos(2\omega_0 t) + \ldots
\]
Fourier Series

- If a function has maximal overlap with one of our cosine functions, then it has zero overlap with all the others!

- We say that our set of cosine functions form an orthogonal basis set...

\[ u_n(t) = \cos(n\omega_0 t) \]

\[ \hat{x}_1 = [0, 1] \]

\[ \hat{x}_2 = [1, 0] \]

\[ \vec{v} = [a_1, a_2] \]

How do we find the coefficients \( a_1 \) and \( a_2 \)?

\[ a_1 = \vec{v} \cdot \hat{x}_1 = \sum_i \vec{v}^i x_1^i \]

\[ a_2 = \vec{v} \cdot \hat{x}_2 = \sum_i \vec{v}^i x_2^i \]
Fourier Series

- Now let’s look at an odd (antisymmetric) function…

\[
y_{odd}(t) = b_1 \sin(\omega_0 t) + b_2 \sin(2\omega_0 t) + b_3 \sin(3\omega_0 t) + \ldots
\]

\[
y_{odd}(t) = \sum_{n=1}^{\infty} b_n \sin(n\omega_0 t)
\]

Why is there no DC term here?
Fourier Series

• For an arbitrary function, we can write it down as the sum of a symmetric and an antisymmetric part.

\[ y(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\omega_0 t) + \sum_{n=1}^{\infty} b_n \sin(n\omega_0 t) \]

- symmetric
- antisymmetric
Learning Objectives for Lecture 11

• Fourier series for symmetric and asymmetric functions

• Complex Fourier series

• Fourier transform

• Discrete Fourier transform (Fast Fourier Transform - FFT)

• Power spectrum
Complex Fourier Series

- We can express any periodic function of time as sums of complex exponentials.

Euler’s formula

\[ e^{i \omega t} = \cos \omega t + i \sin \omega t \]

\[ e^{-i \omega t} = \cos \omega t - i \sin \omega t \]

Rewrite as follows…

\[ \cos \omega t = \frac{1}{2} (e^{i \omega t} + e^{-i \omega t}) \]

\[ \sin \omega t = \frac{1}{2i} (e^{i \omega t} - e^{-i \omega t}) = -\frac{i}{2} (e^{i \omega t} - e^{-i \omega t}) \]

\[ \frac{1}{i} = -i \]
Fourier Series

\[ y(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\omega_0 t) + \sum_{n=1}^{\infty} b_n \sin(n\omega_0 t) \]

\[ y(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \frac{a_n}{2} (e^{in\omega t} + e^{-in\omega t}) + \sum_{n=1}^{\infty} \frac{-ib_n}{2} (e^{in\omega t} - e^{-in\omega t}) \]

\[ y(t) = A_0 + \sum_{n=1}^{\infty} A_n e^{in\omega_0 t} + \sum_{n=1}^{\infty} A_{-n} e^{-in\omega_0 t} \]

‘DC’ or ‘constant’ term

positive frequencies

negative frequencies

\[ A_0 = \frac{a_0}{2} \]
\[ A_n = \frac{1}{2} (a_n - ib_n) \]
\[ A_{-n} = \frac{1}{2} (a_n + ib_n) \]
\[ A_n = (A_{-n})^* \]

complex conjugates
Complex Fourier Series

\[ y(t) = A_0 + \sum_{n=1}^{\infty} A_n e^{in\omega_0 t} + \sum_{n=1}^{\infty} A_{-n} e^{-in\omega_0 t} \]

• We can write this more compactly as follows:

\[ = \sum_{n=0}^{\infty} A_n e^{in\omega_0 t} + \sum_{n=1}^{\infty} A_n e^{in\omega_0 t} + \sum_{n=-1}^{\infty} A_n e^{in\omega_0 t} \]

For \( n = 0 \),

\[ e^{i0} = 1 \]

\[ y(t) = \sum_{n=-\infty}^{\infty} A_n e^{in\omega_0 t} \]
Complex Fourier Series

\[ y(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\omega_0 t) + \sum_{n=1}^{\infty} b_n \sin(n\omega_0 t) \]

- symmetric
- antisymmetric

• We can replace the sine and cosines of the fourier series with a single sum of complex exponentials

\[ y(t) = \sum_{n=-\infty}^{\infty} A_n e^{in\omega_0 t} \]
Complex Fourier Series

- Some examples…

\[ A_{-1} = \frac{1}{2}, \quad A_0 = 0, \quad A_1 = \frac{1}{2} \]

\[ y(t) = \sum_{n=-\infty}^{\infty} A_n e^{i n \omega_0 t} \]

\[ \cos \omega t = \frac{1}{2} \left( e^{i \omega t} + e^{-i \omega t} \right) \]

\[ y(t) = \frac{1}{2} e^{-i \omega_0 t} + \frac{1}{2} e^{i \omega_0 t} = \frac{1}{2} \left( \cos \omega_0 t - i \sin \omega_0 t \right) + \frac{1}{2} \left( \cos \omega_0 t + i \sin \omega_0 t \right) \]

\[ = \cos \omega_0 t \]
Complex Fourier Series

• Some examples...

\[ A_{-2} = \frac{1}{2}, \quad A_0 = 0, \quad A_2 = \frac{1}{2} \]

\[ y(t) = \sum_{n=-\infty}^{\infty} A_n e^{i n \omega_0 t} \]

\[ y(t) = \frac{1}{2} e^{-i 2\omega_0 t} + \frac{1}{2} e^{i 2\omega_0 t} \]

\[ = \frac{1}{2} (\cos 2\omega_0 t - i \sin 2\omega_0 t) + \frac{1}{2} (\cos 2\omega_0 t + i \sin 2\omega_0 t) \]

\[ = \cos 2\omega_0 t \]
Complex Fourier Series

• Some examples...

\[ y(t) = \sum_{n=-\infty}^{\infty} A_n e^{i n \omega_0 t} \]

\[ A_{-2} = \frac{i}{2}, \quad A_0 = 0, \quad A_2 = -\frac{i}{2} \]

\[ y(t) = \frac{i}{2} e^{-i 2 \omega_0 t} + \frac{-i}{2} e^{i 2 \omega_0 t} \]

\[ = \frac{i}{2} (\cos 2 \omega_0 t - i \sin 2 \omega_0 t) + \frac{-i}{2} (\cos 2 \omega_0 t + i \sin 2 \omega_0 t) = \sin 2 \omega_0 t \]
Complex Fourier Series

• The set of functions $e^{in\omega_0 t}$ form an orthogonal basis set over the interval $\left[-\frac{T}{2}, \frac{T}{2}\right]$.

• The $A_0$ coefficient is just the average of our function $y(t)$.

\[
A_0 = \frac{1}{T} \int_{-T/2}^{T/2} y(t) dt = \frac{1}{T} \int_{-T/2}^{T/2} y(t)e^{-0i\omega_0 t} dt
\]

• The $A_1$ coefficient is just the overlap of our function $y(t)$ with $e^{i\omega_0 t}$

\[
A_1 = \frac{1}{T} \int_{-T/2}^{T/2} y(t)e^{-i\omega_0 t} dt
\]

In general

\[
A_m = \frac{1}{T} \int_{-T/2}^{T/2} y(t)e^{-im\omega_0 t} dt
\]

\[
y(t) = \sum_{n=-\infty}^{\infty} A_n e^{in\omega_0 t}
\]
Learning Objectives for Lecture 11

• Fourier series for symmetric and asymmetric functions

• Complex Fourier series

• Fourier transform (I just want you to see this…)

• Discrete Fourier transform (Fast Fourier Transform - FFT)

• Power spectrum
Fourier Transform
(for non-periodic functions)

\[ A_m = \frac{1}{T} \int_{-T/2}^{T/2} y(t) e^{-im\omega_0 t} \, dt \]

\[ y(t) = \sum_{n=-\infty}^{\infty} A_n e^{in\omega_0 t} \]

- We are going to do this by letting the period go to infinity!

\[ T \to \infty \quad , \quad \omega_0 = \frac{2\pi}{T} \to 0 \quad , \quad m\omega_0 \to \omega \quad , \quad A_m \to Y(\omega) \]

\[ Y(\omega) = \int_{-\infty}^{\infty} y(t) e^{-i\omega t} \, dt \]

\[ y(t) = \int_{-\infty}^{\infty} Y(\omega) e^{i\omega t} \, \frac{d\omega}{2\pi} \]
Fourier transform

\[ Y(\omega) = \int_{-\infty}^{\infty} y(t) e^{-i\omega t} \, dt \]
\[ y(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} Y(\omega) e^{i\omega t} \, d\omega \]

- Some examples...

\[ y(t) = 1 \]
\[ Y(\omega) = 2\pi \delta(\omega) \]
\[ y(t) = \int_{-\infty}^{\infty} \delta(\omega) e^{i\omega t} \, d\omega = e^{i0t} = 1 \]
Fourier transform

\[ Y(\omega) = \int_{-\infty}^{\infty} y(t) e^{-i\omega t} \, dt \]

\[ y(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} Y(\omega) e^{i\omega t} \, d\omega \]

- Some examples...

\[ y(t) = e^{i\omega_1 t} \]

\[ Y(\omega) = 2\pi \delta(\omega - \omega_1) \]

\[ y(t) = \int_{-\infty}^{\infty} \delta(\omega - \omega_1) e^{i\omega t} \, d\omega = e^{i\omega_1 t} \]
Fourier transform

\[ Y(\omega) = \int_{-\infty}^{\infty} y(t)e^{-i\omega t} \, dt \]

\[ y(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} Y(\omega)e^{i\omega t} \, d\omega \]

• Some examples...

\[ Y(\omega) = \pi \left[ \delta(\omega + \omega_1) + \delta(\omega - \omega_1) \right] \]

\[ y(t) = \frac{1}{2} e^{-i\omega_1 t} + \frac{1}{2} e^{i\omega_1 t} = \cos \omega_1 t \]
Learning Objectives for Lecture 11

• Fourier series for symmetric and asymmetric functions

• Complex Fourier series

• Fourier transform

• Discrete Fourier transform (Fast Fourier Transform - FFT)

• Power spectrum
Discrete Fourier transform

• Computing the FT and IFT is, in principle really slow

• You have to compute an integral for every value of $\omega$ you want in $Y(\omega)$.

$$Y(\omega) = \int_{-\infty}^{\infty} y(t)e^{-i\omega t} \, dt$$

• It turns out there is a *super fast* computer algorithm called the Fast Fourier Transform (FFT).

First, let’s go back to oscillation frequency $f$, rather than angular frequency $\omega$:

$$f = \frac{\omega}{2\pi}$$

$$Y(f) = \int_{-\infty}^{\infty} y(t)e^{-i2\pi ft} \, dt \quad y(t) = \int_{-\infty}^{\infty} Y(f)e^{i2\pi ft} \, df$$
Discrete Fourier Transform

- Let’s say we have a signal $y(t)$ that is sampled at regular intervals $\Delta t$.

- Let’s say we have $N$ samples, and that $N$ is an even number.

$$F_{samp} = \frac{1}{\Delta t}$$

$$T_{min} = -\frac{N}{2} \Delta t$$

$$T_{max} = \left(\frac{N}{2} - 1\right) \Delta t$$

time$=dt*[-N/2:N/2-1]$;  % time array
Discrete Fourier Transform

• The FFT algorithm returns a discrete Fourier transform that has $N$ frequencies in frequency steps of $\Delta f$

$$\Delta f = \frac{F_{\text{samp}}}{N}$$

$$F_{\text{nyquist}} = \frac{F_{\text{samp}}}{2}$$

freq=df*[-N/2:N/2-1];  % frequency array
Discrete Fourier transform

- One little trick… The FFT algorithm gets the time samples in a strange order, and returns the frequency samples in a strange order…

```
yshft = circshift(y, [0, N/2]);
Y = fft(yshft);

yshft = circshift(Y, [0, N/2]);
```
Discrete Fourier transform

- Some code

```matlab
%%
N=2048; % number of samples in time
%%
dt=.001; % sampling interval
Fs=1./dt; % sampling frequency
time=dt*[-N/2:N/2-1]; % timebase
%
freq=20.; % frequency of sine wave in Hz
y=cos(2*pi*freq*time);
%
%%
yshft=circshift(y,[0,N/2]); % First shift zero point from center to
% first point in the array
ffty=fft(yshft, N)/N; % Now compute the FFT
Y=circshift(ffty,[0,N/2]); % Now shift the spectrum to put zero frequency
% at the middle of the array
%
%Compute the vector of frequencies
df=Fs/N;
Fvec=df*[-N/2:N/2-1];
```
Discrete Fourier transform

- Some examples – sine and cosine

\[ y(t) = \cos(2\pi f_0 t) \quad f_0 = 20\text{Hz} \]

\[ \text{Re}[Y(f)] \quad \text{Im}[Y(f)] \]
Discrete Fourier transform

- Some examples – sine and cosine

\[ y(t) = \sin(2\pi f_0 t) \quad f_0 = 20\text{Hz} \]

![Graph showing continuous sine wave and its Fourier transform](Continuous_sin.m)
Learning Objectives for Lecture 11

• Fourier series for symmetric and asymmetric functions

• Complex Fourier series

• Fourier transform

• Discrete Fourier transform (Fast Fourier Transform - FFT)

• Power spectrum
Introduce idea of ‘Power’

- The electrical power dissipated in a resistor is given by

\[ P(t) = I(t)V(t) = \frac{1}{R}V^2(t) \]

- If the voltage is just a single sine wave at frequency \( \omega \)... \( V(t) = \tilde{V}_\omega \cos(\omega t) \)

\[ V(t) = \tilde{V}_\omega \left[ \frac{1}{2}e^{-i\omega t} + \frac{1}{2}e^{i\omega t} \right] \]

Then the average power from one frequency component is just given by the square magnitude of the F.T. at that frequency...

\[ P(\omega) = \frac{1}{R} |\tilde{V}_\omega|^2 \left( \left| \frac{1}{2}e^{-i\omega t} \right|^2 + \left| \frac{1}{2}e^{i\omega t} \right|^2 \right) = \frac{1}{R} |\tilde{V}_\omega|^2 \left( \left| \frac{1}{2} \right|^2 + \left| \frac{1}{2} \right|^2 \right) = \frac{1}{R} \frac{|\tilde{V}_\omega|^2}{2} \]
Parseval’s Theorem and Power

- The power in each frequency component independently contributes

\[ E = \int_{-\infty}^{\infty} P(t) \, dt = \frac{1}{R} \int_{-\infty}^{\infty} [V(t)]^2 \, dt \]

Parseval’s Theorem says that

\[ \int_{-\infty}^{\infty} [V(t)]^2 \, dt = \int_{-\infty}^{\infty} |\tilde{V}(\omega)|^2 \frac{d\omega}{2\pi} \]

Thus, each frequency component independently contributes to the power in the signal.

It also says that the total variance in the time domain signal is the same as the total variance in the frequency domain signal!
Discrete Fourier transform

- Some examples – sine and cosine

\[ y(t) = \cos(2\pi f_0 t) \quad f_0 = 20\text{Hz} \]
Discrete Fourier transform

• Power spectrum of sine and cosine

For real signals, the power spectrum is symmetric, so only need to plot for positive frequencies!
Discrete Fourier transform

• Some examples – train of delta functions

\[ \Delta T = \text{period} \]

\[ \Delta F = \frac{1}{\Delta T} \]
Discrete Fourier transform

- Power spectrum—train of delta functions

\[ S(f) = |Y(f)|^2 \]

deltafn_train.m
Discrete Fourier transform

- Some examples – square waves
Discrete Fourier transform

- Some examples – square waves

Continuous_square.m

\[ f_0 = 10.75 \text{Hz} \]
Discrete Fourier transform

- Power spectrum—square wave

\[
S(f) = |Y(f)|^2
\]

Spectrum plotted in units of decibels (dB)

\[
10 \log_{10} S(f)
\]
Learning Objectives for Lecture 11

• Fourier series for symmetric and asymmetric functions

• Complex Fourier series

• Fourier transform

• Discrete Fourier transform (Fast Fourier Transform - FFT)

• Power spectrum