9.641 Neural Networks
Problem Set 1
(Due Feb. 10, Thursday before class)

The **integrate-and-fire neuron** is a simple model of spiking behavior that sacrifices biophysical realism for mathematical simplicity.

1. Single neuron model

   First, let’s consider an isolated neuron into which we inject a current \( I_{\text{app}} \). Below threshold, the membrane potential \( V \) obeys the differential equation

   \[
   C \frac{dV}{dt} = -g_L(V - V_L) + I_{\text{app}}
   \]  

   (1)

   If \( V \) reaches a threshold \( V_\theta \), then the neuron is said to spike, and \( V \) is instantaneously reset to a value of \( V_0 \), where \( V_0 < V_\theta \).

   (a) Analytically determine the threshold current \( I_\theta \) (or rheobase) below which the neuron is inactive, and above which the neuron fires repetitively. The sign of \( I_\theta \) should depend on whether \( V_0 \) is above or below \( V_L \).

   (b) Experimentally determine \( I_\theta \) and compare it to the value you found analytically. In MATLAB, a system \( \frac{dy}{dt} = f(y) \) can be simulated by choosing the initial conditions \( y(1) \) and then repeatedly performing the Euler integration step

   \[
   y(t+1) = y(t) + \Delta t \frac{dy}{dt}(t).
   \]

   Use the following values for your simulations: \( V_L = -74mV, g_L = 25nS, V_\theta = -54mV, V_0 = -60mV, C = 500pF \). Plot a trace of the membrane potential \( V \), one for \( I \) right below and one for \( I \) right above \( I_\theta \).

   (c) If \( I_{\text{app}} \) is held constant in time above threshold, the neuron fires action potentials repetitively, as you should have observed in your simulations. Find the relationship between the frequency of firing \( f \) and \( I_{\text{app}} \).

   (d) Show that \( f \) behaves roughly linearly for large \( I_{\text{app}} \) and can be approximated by

   \[
   f \approx \frac{I_{\text{app}} - g_L(V_{1/2} - V_L)}{C(V_\theta - V_0)}
   \]  

   (2)

   with \( V_{1/2} = (V_\theta + V_0)/2 \). Explain in words the reason for this linearity. [**Hint**: Use the Taylor series expansion \( \log(1+z) \approx 1/z + 1/2 \).]

   Plot your results from (c) and (d) together and compare them.

2. Modeling synapses

   A synapse is modeled by a variable conductance \( g \) in the postsynaptic neuron. A spike in the presynaptic neuron causes an increase of the conductance according
to \( g := g + \frac{a}{\tau} \). Between spikes, \( g \) decays exponentially: \( \frac{dg}{dt} = -\frac{g}{\tau} \). So a synapse is a leaky integrator, counting spikes but forgetting them over time periods longer than \( \tau \). The area under the exponential caused by a single spike is given by the parameter \( \alpha \).

Under certain conditions this can be approximated by \( \tau \frac{dx}{dt} + x \approx f \), where \( x \) is proportional to \( g \) and \( f \) is the frequency of incoming spikes.

Simulate the time course of the conductance of a synapse for \( f = 25 \) Hz for different \( \tau \). For what values of \( \tau \) is this approximation valid? Illustrate your answer with two plots.

3. From synapses to current

In practice, neurons are a part of networks and receive input currents through synapses instead of an electrode. For a neuron \( i \) receiving inputs from neurons \( j \), this can be written as:

\[
C_i \frac{dV_i}{dt} = -g_{Li}(V_i - V_L) - \sum_j g_{ij}(V_i - V_{ij})
\]

(3)

Show that equation (3) can be simplified to the form of equation (1), describing a neuron with leak conductance \( g_L \) receiving an external current \( I_{app} \) if the synaptic conductances \( g_{ij} \) are changing slowly (meaning they are constant for a small interval \( dt \)). Determine \( I_{app} \) and \( g_L \) analytically in terms of \( g_{ij}, V_{ij}, V_L \) and \( g_{Li} \).

4. From spikes to rates

We are now ready to derive a nonspiking model of a neuron. To do that, we will assume that all neurons have the same membrane capacitance \( C \), the same time constant \( \tau \) and that conductances are changing slowly (meaning they are constant for a small interval \( dt \)).

Using the results of 1, 2 and 3, show that equation (1) can be approximated by

\[
\tau \frac{dx_i}{dt} + x_i \approx f \left( b_i + \sum_j W_{ij} x_j \right)
\]

(4)

Starting with the approximation in (2), plug in the approximated f-I relationship from 1(d). Then, substitute \( I_{app} \) and \( g_L \) with the expressions you found in (3). Assuming all time constants are the same, all synapses emanating from a single neuron have the same temporal behavior, because they are driven by the same spike train, and decay at the same rate. This yields \( x_j = \frac{b_j}{\alpha_{ij}} \). Finally, identify \( b_i \) and \( W_{ij} \) in terms of \( \alpha_{ij}, g_{Li}, V_L, V_{1/2} \) and \( V_{ij} \).