Problem 1. Consider the matrix \( A = \begin{bmatrix} 2 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 2 \end{bmatrix} \)

(a) Compute the product \( Av \) for the vector \( v = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \).

Answer

Using the rule \((Av)_j = \sum_{k=1}^{N} a_{jk}v_k\), we have

\[
Av = \begin{bmatrix} 2 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} [(2)(1) + (1)(2) + (-1)(3)] \\ [(1)(1) + (3)(2) + (1)(3)] \\ [(-1)(1) + (1)(2) + (2)(3)] \end{bmatrix} = \begin{bmatrix} 1 \\ 10 \\ 7 \end{bmatrix}
\]

(b) Compute the LU decomposition \( A = LU \) where \( L \) is a lower-triangular matrix and \( U \) is an upper-triangular matrix.

Answer

To generate \( A = LU \) we perform Gaussian elimination without partial pivoting. First, we zero the (2,1) component, calculating

\[
\lambda_{21} = \frac{a_{21}}{a_{11}} = \frac{1}{2} = 0.5
\]

and then by performing the row operation \( 2 \leftarrow 2 - \lambda_{21} \times 1 \), we have the new matrix

\[
A^{(2,1)} = \begin{bmatrix} 2 & 1 & -1 \\ [1 - (0.5)(2)] & [3 - (0.5)(1)] & [1 - (0.5)(-1)] \\ [-1 & 1 & 2] \end{bmatrix} = \begin{bmatrix} 2 & 1 & -1 \\ 0 & 2.5 & 1.5 \\ -1 & 1 & 2 \end{bmatrix}
\]
Next, we zero the (3,1) component by calculating \( \lambda_{31} = \frac{a_{31}^{(2,1)}}{a_{11}^{(2,1)}} = -\frac{1}{2} = -0.5 \) and performing the row operation on \( A^{(2,1)} \rightarrow 3 \leftarrow 3 - \lambda_{31} \times 1 \), to yield

\[
A^{(3,1)} = \begin{bmatrix}
2 & 1 & -1 \\
0 & 2.5 & 1.5 \\
[-1 - (-0.5)(2)] [1 - (-0.5)(1)] [2 - (-0.5)(-1)]
\end{bmatrix} = \begin{bmatrix}
2 & 1 & -1 \\
0 & 2.5 & 1.5 \\
0 & 1.5 & 1.5
\end{bmatrix}
\]

Next, we zero the (3,2) component by calculating \( \lambda_{32} = \frac{a_{32}^{(3,1)}}{a_{22}^{(3,1)}} = \frac{1.5}{2.5} = \frac{3}{5} = 0.6 \) and performing the row operation on \( A^{(3,1)} \rightarrow 3 \leftarrow 3 - \lambda_{32} \times 2 \), to yield the upper triangular matrix

\[
U = \begin{bmatrix}
2 & 1 & -1 \\
0 & 2.5 & 1.5 \\
0 & [1.5 - (0.6)(2.5)] [1.5 - (0.6)(1.5)]
\end{bmatrix} = \begin{bmatrix}
2 & 1 & -1 \\
0 & 2.5 & 1.5 \\
0 & 0 & 0.6
\end{bmatrix}
\]

To generate the lower triangular matrix \( L \), we place ones along the principal diagonal and store the values of the \( \lambda_{mn} \)'s below the diagonal,

\[
L = \begin{bmatrix}
1 & 0 & 0 \\
\lambda_{21} & 1 & 0 \\
\lambda_{31} & \lambda_{32} & 1
\end{bmatrix} = \begin{bmatrix}
1 & 0 & 0 \\
0.5 & 1 & 0 \\
-0.5 & 0.6 & 1
\end{bmatrix}
\]

(c) Compute the determinant of \( A \).

**Answer**

Here, we save some time by noting that \( |A| = |LU| = |L||U| \), and remembering that the determinant of a triangular matrix is the product of its diagonal elements. Therefore, \( |L| = 1 \) and

\[
|A| = |U| = U_{11} U_{22} U_{33} = (2)(2.5)(0.6) = 3
\]

(d) Compute the solution \( \vec{x} \) to the linear system \( A \vec{x} = \vec{b} \) for \( \vec{b} = \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix} \).

**Answer**
We can compute the solution quickly by using the LU factorization. Substituting for $A$,

$$LUx = b \quad \Rightarrow \quad Lc = b \quad Ux = c$$

First, we compute $c$ by solving $Lc = b$ through forward substitution,

$$\begin{bmatrix} 1 & 0 & 0 \\ 0.5 & 1 & 0 \\ -0.5 & 0.6 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix} \quad \Rightarrow \quad (0.5)c_1 + (1)c_2 = 0 \quad c_2 = -0.5(c_1) = -1 \quad (-0.5)c_1 + (0.6)c_2 + c_3 = -1 \quad c_3 = -1 - 0.6c_2 + 0.5c_1 = 0.6$$

Then, we solve $Ux = c$ through backward substitution,

$$\begin{bmatrix} 2 & 1 & -1 \\ 0.5 & 1.5 & 0 \\ 0 & 0.6 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 0.6 \end{bmatrix} \quad \Rightarrow \quad (2)x_1 + (1)x_2 + (-1)x_3 = 2 \quad x_1 = \frac{2 + 1 + 1}{2} = 2 \quad (2.5)x_2 + (1.5)x_3 = -1 \quad x_2 = \frac{-1 - (1.5)(1)}{2.5} = -1 \quad (0.6)x_3 = 0.6 \quad x_3 = 1$$

Therefore, the solution is $x = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$.

**Problem 2.** Consider the system of three nonlinear algebraic equations,

$$f_1(x) = x_1^2 + x_2 - \frac{1}{2}x_3^2 = 0$$

$$f_2(x) = x_1 + x_2^3 + \frac{1}{3}x_3^3 = 0$$

$$f_3(x) = -\frac{1}{2}x_1^2 + x_2 + x_3^2 = 0$$

(a) Compute the Jacobian matrix, where the elements are functions of $x$.

**Answer**

The Jacobian matrix $J(x)$ has elements $J_{mn} = \frac{\partial f_m}{\partial x_n}$. Thus, for this system we have
\[ J(\mathbf{x}) = \begin{bmatrix} 2x_1 & 1 & -x_3 \\ 1 & 3x_2^2 & x_3^2 \\ -x_1 & 1 & 2x_3 \end{bmatrix} \]

(b) Using \( x^{[0]} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \), compute the new estimate \( x^{[1]} \) generated by Newton’s method. \textit{Hint:}

Does \( J(x^{[0]}) \) look familiar?

\textbf{Answer}

The linear system that we wish to solve is \( J(x^{[0]})p^{[0]} = -f(x^{[0]}) \). For this particular \( x^{[0]} \), the Jacobian is equal to the matrix \( A \) from problem 1,

\[ J(x^{[0]}) = \begin{bmatrix} 2 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 2 \end{bmatrix} \]

This means that we can use the LU decomposition from problem 1 to avoid performing Gaussian elimination again. The function vector is

\[ f(x^{[0]}) = \begin{bmatrix} x_1^2 + x_2 + 1 - \frac{1}{2}x_3 \\ x_1 + x_2^3 + \frac{1}{3}x_3^3 \\ -\frac{1}{2}x_2^3 + x_2 + x_3^2 \end{bmatrix} = \begin{bmatrix} 1 + 1 - \frac{1}{2} \\ 1 + 1 + \frac{1}{3} \\ -\frac{1}{2} + 1 + 1 \end{bmatrix} = \begin{bmatrix} 1.5 \\ 2.333 \\ 1.5 \end{bmatrix} \]

The linear system that we wish to solve is of the form \( Ax = b \) with

\[ b = -f(x^{[0]}) = \begin{bmatrix} -1.5 \\ -2.333 \\ -1.5 \end{bmatrix} \]

Repeating the forward and backward substitution process of (1.d), we have a full Newton-update vector and new solution estimate,

\[ p^{[0]} = \begin{bmatrix} -2.167 \\ 0.667 \\ -2.167 \end{bmatrix}, \quad x^{[1]} = x^{[0]} + p^{[0]} = \begin{bmatrix} -1.167 \\ 1.667 \\ -1.167 \end{bmatrix} \]

(c) Would this guess be accepted in a robust reduced-step Newton algorithm?

\textbf{Answer}
The square of the 2-norm (length) of the function vector at $x^{[10]}$ is

$$\left| f(x^{[10]}) \right|^2 = f(x^{[10]}) \cdot f(x^{[10]}) = 9.9429$$

For the new estimate, $f(x^{[11]}) = \begin{bmatrix} 2.379 \\ 2.3479 \end{bmatrix}$ and thus

$$\left| f(x^{[11]}) \right|^2 = f(x^{[11]}) \cdot f(x^{[11]}) = 19.6436$$

As $\left| f(x^{[11]}) \right|^2 > \left| f(x^{[10]}) \right|^2$, we would not accept this new estimate, but would rather iteratively halve the step length until we find that the 2-norm is reduced at the new estimate.

**Problem 3.** Consider again the matrix $A$ of problem 1.

(a) What properties of the eigenvalues and eigenvectors of $A$ can you infer simply by inspection of $A$, i.e. with no additional computations?

**Answer**

Since $A$ is real-symmetric, we know that its eigenvalues are all real and its eigenvectors are mutually orthogonal.

(b) Compute an upper bound on the largest possible magnitude (modulus) of an eigenvalue of $A$. That is, find a value $\lambda_{\text{max}}$ such that for all eigenvalues $\lambda_j$ of $A$, we are guaranteed to have $|\lambda_j| \leq \lambda_{\text{max}}$.

**Answer**

Here, we use Gershgorin’s theorem and the fact that we know all eigenvalues of $A$ to be real. Gershgorin’s theorem states that for each $\lambda$, such that $A \omega = \lambda \omega$, one of the following inequalities must apply,

$$|\lambda - 2| \leq |1| + |1| = 2 \quad \Rightarrow \quad 0 \leq \lambda \leq 4$$
$$|\lambda - 3| \leq |1| + |1| = 2 \quad \Rightarrow \quad 1 \leq \lambda \leq 5$$
$$|\lambda - 2| \leq |1| + |1| = 2 \quad \Rightarrow \quad 0 \leq \lambda \leq 4$$

Thus, we have $0 \leq \lambda \leq 5$, so that all eigenvalues must be non-negative and must be less than or equal to 5. 5 is thus an upper bound on the magnitude of the eigenvalues of $A$. In fact, the eigenvalues are 0.2679, 3.0000, 3.7321.