1.1.1

EXPRESSING SYSTEMS OF LINEAR ALGEBRAIC EQUATIONS AS: \( A \mathbf{x} = \mathbf{b} \)

We wish to solve systems of simultaneous linear algebraic equations of the general form:

\[
\begin{align*}
  a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\
  a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2 \\
  \vdots &\quad \vdots \\
  a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n &= b_n
\end{align*}
\]

(1.1.1-1)

Where we have \( N \) equations for the \( N \) unknowns \( x_1, x_2, \ldots, x_n \).

As a particular example, consider the following set of these three equations (\( N=3 \)) for the three unknowns \( x_1, x_2, x_3 \):

\[
\begin{align*}
  x_1 + x_2 + x_3 &= 4 \\
  2x_1 + x_2 + 3x_3 &= 7 \\
  3x_1 + x_2 + 6x_3 &= 2
\end{align*}
\]

(1.1.1-2)

\( a_{ij} = \) constant coefficient (usually real) multiplying unknown \( x_j \) in equation \( #i \).
\( b_i = \) constant “right-hand-side” coefficient for equation \( #i \).

For the system (1.1.1-2) above,

\[
\begin{align*}
  a_{11} &= 1 & a_{12} &= 1 & a_{13} &= 1 & b_1 &= 4 \\
  a_{21} &= 2 & a_{22} &= 1 & a_{23} &= 3 & b_2 &= 7 \\
  a_{31} &= 3 & a_{32} &= 1 & a_{33} &= 6 & b_3 &= 2
\end{align*}
\]
It is common to write linear systems in matrix/vector for as:

\[ A \mathbf{x} = \mathbf{b} \quad (1.1.1-3) \]

Where the vector of unknowns \( \mathbf{x} \) is written as:

\[
\mathbf{x} = \begin{bmatrix}
    x_1 \\
    x_2 \\
    \vdots \\
    x_n
\end{bmatrix} \quad (1.1.1-4)
\]

The vector of right-hand-side coefficients \( \mathbf{b} \) is written:

\[
\mathbf{b} = \begin{bmatrix}
    b_1 \\
    b_2 \\
    \vdots \\
    b_n
\end{bmatrix} \quad (1.1.1-5)
\]

The matrix of coefficients \( A \) is written in a form with \( N \) rows and \( N \) columns,

\[
A = \begin{bmatrix}
    a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\
    a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\
    \vdots & \vdots & \vdots & \ddots & \vdots \\
    a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn}
\end{bmatrix} \quad (1.1.1-6)
\]

We see that row ‘i’ contains the values \( a_{i1}, a_{i2}, \ldots, a_{iN} \) that are the coefficients multiplying each unknown \( x_1, x_2, \ldots, x_N \) in equation \( #i \).

Rows \( \Leftrightarrow \) equations \quad Columns \( \Leftrightarrow \) coefficients multiplying a specific unknown in each equation.

\( a_{ij} = \) element of \( A \) in ith row and jth column
\( = \) coefficient multiplying \( x_j \) in equation \( #i \).
After we will write the coefficients in matrix form explicitly, so that we may write $A \mathbf{x} = \mathbf{b}$ as:

$$
\begin{bmatrix}
    a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\
    a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\
    \vdots & \vdots & \vdots & \ddots & \vdots \\
    a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn}
\end{bmatrix}
\begin{bmatrix}
    x_1 \\
    x_2 \\
    \vdots \\
    x_N
\end{bmatrix}
= 
\begin{bmatrix}
    b_1 \\
    b_2 \\
    \vdots \\
    b_n
\end{bmatrix}
$$

(1.1.1-7)

For the example system (1.1.1-2):

$$
\begin{align*}
x_1 + x_2 + x_3 &= 4 \\
2x_1 + x_2 + 3x_3 &= 7 \\
3x_1 + x_2 + 6x_3 &= 2
\end{align*}
$$

(1.1.1-2, repeated)

We have:

$$
A = 
\begin{bmatrix}
    1 & 1 & 1 \\
    2 & 1 & 3 \\
    3 & 1 & 6
\end{bmatrix}
\quad \quad \quad 
\mathbf{b} = 
\begin{bmatrix}
    4 \\
    7 \\
    2
\end{bmatrix}
$$

(1.1.1-8)

As we will represent our linear systems as matrices “acting on” vectors, some review of basic vector notation is required.