1.2.2 Gauss-Jordan Elimination

In the method of Gaussian elimination, starting from a system $A \mathbf{x} = \mathbf{b}$ of the general form

\[
\begin{bmatrix}
  a_{11} & a_{12} & a_{13} & \ldots & a_{1n} \\
  a_{21} & a_{22} & a_{23} & \ldots & a_{2n} \\
  \vdots & \vdots & \vdots & \ddots & \vdots \\
  a_{n1} & a_{n2} & a_{n3} & \ldots & a_{nn}
\end{bmatrix}
\begin{bmatrix}
  x_1 \\
  x_2 \\
  \vdots \\
  x_n
\end{bmatrix}
= 
\begin{bmatrix}
  b_1 \\
  b_2 \\
  \vdots \\
  b_n
\end{bmatrix}
\]

(1.2.2-1)

is converted to an equivalent system $A' \mathbf{x} = \mathbf{b'}$ after $\frac{2}{3}N^3$ FLOP’s that is of upper triangular form

\[
\begin{bmatrix}
  a'_{11} & a'_{12} & a'_{13} & \ldots & a'_{1N} \\
  a'_{22} & a'_{23} & a'_{24} & \ldots & a'_{2N} \\
  \vdots & \vdots & \vdots & \ddots & \vdots \\
  a'_{NN} & \end{bmatrix}
\begin{bmatrix}
  x_1 \\
  x_2 \\
  \vdots \\
  x_N
\end{bmatrix}
= 
\begin{bmatrix}
  b'_1 \\
  b'_2 \\
  \vdots \\
  b'_N
\end{bmatrix}
\]

(1.2.2-2)

At this point, it is possible, through backward substitution, to solve for the unknowns in the order $x_N, x_{N-1}, x_{N-2}, \ldots$ in $\frac{N^2}{2}$ steps.

In the method of Gauss-Jordan elimination, one continues the work of elimination, placing zeros above the diagonal. To “zero” the element at $(N-1, N)$, we write the last two equations of (1.2.2-2)

\[
\begin{align*}
  a'_{N-1,N-1}x_{N-1} + a'_{N-1,N}x_N &= b'_{N-1} \\
  a'_{NN}x_N &= b'_N
\end{align*}
\]

(1.2.2-3)
We then define \( \lambda_{N-1,N} = \frac{a_{N-1,N}'}{a_{NN}} \) (1.2.2-4)

And replace the N-1st row with the equation obtained after performing the row operation

\[
\begin{align*}
(a_{N-1,N-1}' x_{N-1} + a_{N-1,N}' x_N &= b_{N-1}') \\
-\lambda_{N-1,N} (a_{NN}' x_N &= b_N') \\
\end{align*}
\]

Defining

\[
b_{N-1}'' = b_{N-1}' - b_N' \lambda_{N-1,N} \tag{1.2.2-6}
\]

and noting

\[
a_{N-1}' = a_{N-1,N}' - \lambda_{N-1,N} a_{NN}' = a_{N-1,N}' - \left(\frac{a_{N-1,N}'}{a_{NN}}\right) a_{NN}' = 0 \tag{1.2.2-7}
\]

After this row operation the set of equations becomes

\[
\begin{bmatrix}
a_{11} & a_{12} & a_{13} & \ldots & a_{1,N-1}' & a_{1,N}' \\
a_{22} & a_{23} & \ldots & a_{2,N-1}' & a_{2,N}' \\
a_{33} & \ldots & a_{3,N-1}' & a_{3,N}' \\
\vdots & & \vdots & & \vdots \\
a_{N-1,N-1}' & \ldots & 0 & a_{NN}'
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
\vdots \\
x_{N-1} \\
x_N
\end{bmatrix}
= \begin{bmatrix}
b_1' \\
b_2' \\
b_3' \\
\vdots \\
b_{N-1}' \\
b_N'
\end{bmatrix} \tag{1.2.2-8}
\]

We can continue this process until the set of equations is in diagonal form

\[
\begin{bmatrix}
a_{11}' \\
a_{22}' \\
a_{33}' \\
\vdots \\
a_{NN}'
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
\vdots \\
x_N
\end{bmatrix}
= \begin{bmatrix}
b_1'' \\
b_2'' \\
b_3'' \\
\vdots \\
b_N''
\end{bmatrix} \tag{1.2.2-9}
\]
Dividing each equation by the value of its single coefficient yields

\[
\begin{bmatrix}
1 & x_1 \\
1 & x_2 \\
1 & x_3 \\
\vdots & \vdots \\
1 & x_N
\end{bmatrix} =
\begin{bmatrix}
b_1 \\
a_{11} \\
b_2 \\
a_{22} \\
b_3 \\
a_{33} \\
\vdots \\
b_N \\
a_{NN}
\end{bmatrix}
\begin{bmatrix}
a_{11} \\
a_{22} \\
a_{33} \\
\vdots \\
a_{NN}
\end{bmatrix}
\]

(1.2.2-10)

The matrix on the left that has a one everywhere along the principal diagonal and zeros everywhere else is called the identity matrix, and has the property that for any vector \( \mathbf{y} \),

\[ I\mathbf{y} = \mathbf{y} \quad (1.2.2-11) \]

The form (1.2.2-10) therefore immediately gives the solution to the problem.

In practice, we use Gaussian Elimination, stopping at (1.2.2-2) to begin backward substitution rather than continue the elimination process because backward substitution is so fast, \( N^2 \ll 2N^3/3 \) for all but small problems.

We therefore do not consider the method of Gauss-Jordan Elimination further.