1.3.4 Null Space (kernel) and Existence/Uniqueness of Solutions

We now have the tools necessary to consider the existence and uniqueness of solutions to the linear system of equations

\[ \mathbf{A} \mathbf{x} = \mathbf{b} \quad (1.3.4-1) \]

Where \( \mathbf{x}, \mathbf{b} \in \mathbb{R}^N \) and \( \mathbf{A} \) is a \( N \times N \) real matrix.

As described in section 1.3.1, we interpret \( \mathbf{A} \) as a linear transformation that maps each \( \mathbf{v} \in \mathbb{R}^N \) into some \( \mathbf{A} \mathbf{v} \in \mathbb{R}^N \) according to the rule

\[
\mathbf{A} \mathbf{v} = \begin{bmatrix}
  a_{11} & a_{12} & \cdots & a_{1N} \\
  a_{21} & a_{22} & \cdots & a_{2N} \\
  \vdots & \vdots & & \vdots \\
  a_{N1} & a_{N2} & \cdots & a_{NN}
\end{bmatrix}
\begin{bmatrix}
  v_1 \\
  v_{12} \\
  \vdots \\
  v_N
\end{bmatrix}
= \begin{bmatrix}
  a_{11}v_1 + a_{12}v_{12} + \cdots + a_{1N}v_N \\
  a_{21}v_1 + a_{22}v_{12} + \cdots + a_{2N}v_N \\
  \vdots \\
  a_{N1}v_1 + a_{N2}v_{12} + \cdots + a_{NN}v_N
\end{bmatrix} \quad (1.3.4-2)
\]

Pictorially, we view the problem of solving \( \mathbf{A} \mathbf{x} = \mathbf{b} \) as finding the (or one of many?) vector(s) \( \mathbf{x} \in \mathbb{R}^N \) that maps into a specific \( \mathbf{b} \) under \( \mathbf{A} \).

Here we have shown that for any real matrix \( \mathbf{A} \), the rule for forming \( \mathbf{A} \mathbf{v} \) (1.3.4-3) guarantees that

\[ \mathbf{A} \mathbf{0} = \mathbf{0} \quad (1.3.4-4) \]

Where \( \mathbf{0} \) is the null vector, \( \mathbf{0} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \) \( (1.3.4-5) \)
We always have one vector, \( \mathbf{0} \), that maps into \( \mathbf{0} \) under \( A \). Crucial to the question of existence and uniqueness of solutions is the existence of any other vectors \( \mathbf{w} \neq \mathbf{0} \) that also map into \( \mathbf{0} \) under \( A \).

We define the null space (or kernel) of a real matrix \( A \) to be the set of all vectors \( \mathbf{w} \in \mathbb{R}^N \) such that \( A\mathbf{w} = \mathbf{0} \). Pictorially, we view the kernel of \( A \), denoted \( \text{K}_A \), as

We use the concept of the kernel (null space) to prove the following theorems on existence/uniqueness of solutions to \( A\mathbf{x} = \mathbf{b} \).

**Theorem 1.3.4.1**

Let \( \mathbf{x} \in \mathbb{R}^N \) be a solution to the linear system \( A\mathbf{x} = \mathbf{b} \), where \( \mathbf{b} \in \mathbb{R}^N \), \( A \) is an \( N \times N \) real matrix. If the kernel of \( A \) contains only the null vector, i.e. \( \text{K}_A = \{\mathbf{0}\} \), then this solution is unique (no other solutions exist).
Proof:

Let \( \mathbf{x} \) satisfy \( A\mathbf{x} = \mathbf{b} \). Let \( \mathbf{y} \) be some vector in \( \mathbb{R}^N \) that also satisfies the system of equations \( A\mathbf{y} = \mathbf{b} \).

If we define \( \mathbf{v} = \mathbf{y} - \mathbf{x} \), we can write this 2\(^{nd}\) solution as

\[
\mathbf{y} = \mathbf{x} + \mathbf{v} \tag{1.3.4-6}
\]

Then,

\[
A\mathbf{y} = A(\mathbf{x} + \mathbf{v}) = A\mathbf{x} + A\mathbf{v} \tag{1.3.4-7}
\]

Since \( \mathbf{x} \) is a solution, \( A\mathbf{x} = \mathbf{b} \), and

\[
A\mathbf{y} = \mathbf{b} + A\mathbf{v} \tag{1.3.4-8}
\]

If \( \mathbf{y} \) is to be a solution as well, then \( A\mathbf{y} = \mathbf{b} \). This can then be the case only if

\[
A\mathbf{v} = 0 \tag{1.3.4-9}
\]

Therefore, if \( \mathbf{x} \) is a solution, every other solution must differ from \( \mathbf{x} \) by a vector \( \mathbf{v} \in \text{K}_A \).

Since we have stated that for our matrix \( A \), the only vector in the kernel is the null vector \( \mathbf{0} \), there are no other solutions \( \mathbf{y} \neq \mathbf{x} \) to \( A\mathbf{x} = \mathbf{b} \).

Q.E.D. = “Quod Erat Demonstrandum”

“That which was to have been proven”

We have proven a theorem on uniqueness. We must not prove a theorem on existence.
To do so, we define the range of $A$, denoted $R_A$, to be the subset of all vectors $y \in \mathbb{R}^N$ such that there exists some $v \in \mathbb{R}^N$ with $Av = y$.

There exists some vector $y$

$$R_A \equiv \left\{ y \in \mathbb{R}^N \mid \exists v \in \mathbb{R}^N \text{ with } Ay = y \right\} \quad (1.3.4-10)$$

Every $y \in \mathbb{R}^N$ under the condition that

Pictorially, we view the range as

No vectors map into the port of $\mathbb{R}^N$ outside of the range.

**Theorem 1.3.4.2:**

Let $A$ be a real $N \times N$ matrix with kernel $K_A \subseteq \mathbb{R}^N$ and Range $R_A \subseteq \mathbb{R}^N$. Then

(I) the dimensions of the kernel and of the range satisfy the “dimension theorem”

$$\dim(K_A) + \dim(R_A) = N \quad (1.3.4-11)$$

(II) If the kernel contains only the null vector $0$, $\dim(K_A) = 0$. As the range therefore has dimension $N$, $R_A = \mathbb{R}^N$, and for every $b \in \mathbb{R}^N$, there exists some $x \in \mathbb{R}^N$ with $Ax = b$ (existence).
Proof:

(I) Let us use an orthonormal basis \{U^{[1]}, U^{[2]}, \ldots, U^{[M]}, U^{[M+1]}, \ldots, U^{[N]}\} for \(\mathbb{R}^N\) such that the 1st \(M\) vectors form a basis for the kernel \(K_A\).

Since the kernel satisfies all the properties of a vector space itself, we can construct the \(M\) basis vectors for \(K_A\), for example by Gram-Schmidt orthogonalization. Once we have identified these \(M\) basis vectors, we can continue with Gram-Schmidt orthogonalization to finish the basis set.

We can therefore write any \(w \in K_A\) as

\[
W = c_1 U^{[1]} + c_2 U^{[2]} + \ldots + c_M U^{[M]} \quad (1.3.4-12)
\]

And the dimension of the kernel is obviously \(M\),

\[
\text{dim}(K_A) = M \quad (1.3.4-13)
\]

We now write any arbitrary vector \(v \in \mathbb{R}^N\) as an expansion in the basis,

\[
v = v_1 U^{[1]} + v_2 U^{[2]} + \ldots + v_M U^{[M]} + v_{M+1} U^{[M+1]} + \ldots + v_N U^{[N]} \quad (1.3.4-14)
\]

Then, taking the product with \(A\),

\[
A v = A \left( v_1 U^{[1]} + v_2 U^{[2]} + \ldots + v_M U^{[M]} \right) + v_{M+1} A U^{[M+1]} + \ldots + v_N A U^{[N]} \quad (1.3.4-15)
\]

We therefore see that any vector \(A v \subseteq \mathbb{R}_A\) can be written as a linear combination of the \(N - M\) vectors \(\{A U^{[M+1]}, \ldots, A U^{[N]}\}\).

Therefore \(\text{dim}(R_A) = N - M\) and \(\text{dim}(K_A) + \text{dim}(R_A) = N\)

(II) Follows directly
Taken jointly, theorems 1.3.4.1 and 1.3.4.2 demonstrate that if $K_A = 0$, i.e. only the null vector maps into the null vector under $A$, then $Ax = b$ has a unique solution for all $b$.

What happens if the kernel of $A$ is not empty, i.e. there exists some $w \neq 0$? Let us consider a specific example.

Look at a system with

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (1.3.4-16)$$

Then for any $v \in \mathbb{R}^3$

$$Av = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ v_3 \end{bmatrix} \quad (1.3.4-17)$$

Writing

$$v = v_1 e^{[1]} + v_2 e^{[2]} + v_3 e^{[3]}, \quad (1.3.4-18)$$

$$Av = v_1 Ae^{[1]} + v_2 Ae^{[2]} + v_3 Ae^{[3]} \quad (1.3.4-19)$$

With

$$Ae^{[1]} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = 0$$

$$Ae^{[2]} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = 0$$

$$Ae^{[3]} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = e^{[3]} \quad (1.3.4-20)$$

Therefore

$$Av = v_1 0 + v_2 0 + v_3 e^{[3]} = v_3 e^{[3]} \quad (1.3.4-21)$$

This information is “lost” when mapped by $A$
We therefore see that for this $A$, any vector that is a linear combination of $e^{[1]}$ and $e^{[2]}$ is part of the kernel,

$$w = w_1e^{[1]} + w_2e^{[2]} \in K_A \quad (1.3.4-22)$$

we then can say that $K_A = \text{span}\{e^{[1]}, e^{[2]}\}$, and so $\text{dim}(K_A) = z. \quad (1.3.4-23)$

Also since for any $v \in \mathbb{R}^3$, $A\bar{v} = \begin{bmatrix} 0 \\ 0 \\ v_3 \end{bmatrix} = v_3e^{[3]}$; therefore $R_A = \text{span}\{e^{[3]}\}$, $\text{dim}(R_A) = 1 \quad (1.3.4-24)$

As expected from the dimension theorem, $\text{dim}(K_A) + \text{dim}(R_A) = 3 \quad (1.3.4-25)$

Now, does $A\bar{x} = \bar{b}$ have a solution?

- if $\bar{b} \in R_A$, i.e. $\bar{b} = \begin{bmatrix} 0 \\ 0 \\ b_3 \end{bmatrix} \quad (1.3.4-26)$, then yes, there is a solution.

We easily see that a solution is

$$\bar{x} = \begin{bmatrix} 0 \\ 0 \\ b_3 \end{bmatrix} \quad (1.3.4-27), \quad A\bar{x} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ b_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ b_3 \end{bmatrix} = \bar{b} \quad (1.3.4-28)$$

There are however an infinite number of solutions, since any vector $\bar{x} + w_1e^{[1]} + w_2e^{[2]}$ is also a solution as

$$A(\bar{x} + w_1e^{[1]} + w_2e^{[2]}) = A\bar{x} + w_1Ag^{[1]} + w_2Ag^{[2]}$$

$$= A\bar{x} + w_10 + w_20R_A \quad (1.3.4-29)$$

- if $\bar{b} \notin R_A$, i.e. $\bar{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ with either $b_1 \neq 0$ or $b_2 \neq 0$, then $A\bar{x} = \bar{b}$ has no solution.
We see therefore that we have the following three possibilities regarding the existence and uniqueness of solutions to the linear system $A\mathbf{x} = \mathbf{b}$, $A$ an $N \times N$ real matrix, $\mathbf{b} \in \mathbb{R}^N$.

**Case I**

The kernel of $A$ is empty, i.e. $K_A = \{0\}$. Then, $R_A = \mathbb{R}^N$ and for all $\mathbf{b} \in \mathbb{R}^N$ there exists a unique solution $\mathbf{x}$.

**Case II**

There exists $\mathbf{w} \neq 0$ for which $A\mathbf{w} = 0$. Let $\dim(K_A) = M$, and $\{U^{[1]}, U^{[2]}, \ldots, U^{[M]}\}$ forms an orthonormal basis $K_A$,

$$
\mathbf{w} = c_1U^{[1]} + c_2U^{[2]} + \ldots + c_MU^{[M]} \in K_A, \quad A\mathbf{w} = 0 \quad (1.3.4-30)
$$

If then $b \cdot U^{[1]} = b \cdot U^{[2]} = \ldots = b \cdot U^{[M]} = 0$, then $\mathbf{b} \in R_A$ and solutions exist, but there are an infinite number. If $A\mathbf{x} = \mathbf{b}$, then $A(\mathbf{x} + c_1U^{[1]} + \ldots + c_MU^{[M]}) = \mathbf{b}$ \hspace{1cm} (1.3.4-31) as well.

**Case III**

Again $\dim(K_A) = M$, $M \geq 1$ and $\{U^{[1]}, \ldots, U^{[M]}\}$ forms an orthonormal basis for $K_A$.

Now, for at least one $U^{[j]}$, $j = 1, 2, \ldots, M$, $b \cdot U^{[j]} \neq 0$. Therefore $\mathbf{b} \notin R_A$ and the system $A\mathbf{x} = \mathbf{b}$ has no solution.

While these rules provide insight into existence and uniqueness, to employ them we need:

1. A method to determine if $K_A = \{0\}$ from the coefficients of $A$
2. A method to identify basis vectors for $K_A$

Point (1) is the subject of the next section. (2) is discussed in context of eigenvalues.