**PRINCIPLES OF QUANTUM MECHANICS (cont’d)**

**COMMUTATORS**

Order counts when applying multiple operators!

*Example:* \( \hat{A} = \hat{p}\hat{x} \) ? and can we write \( \hat{p}\hat{x} = \hat{x}\hat{p} \)?

\[
\Rightarrow \quad \text{operate on function to obtain } \quad \hat{A} f(x) = g(x) \quad \Rightarrow \quad (\hat{p}\hat{x}) f(x) = \left( -i\hbar \frac{d}{dx} \right) f(x) \\
= -i\hbar x \frac{d}{dx} f(x) - i\hbar f(x) = \left( -i\hbar x \frac{d}{dx} - i\hbar \right) f(x) \\
\therefore \quad \hat{A} = \left( -i\hbar x \frac{d}{dx} - i\hbar \right) = (\hat{p}\hat{x})
\]

Now try \( \hat{B} = \hat{x}\hat{p} \)

\[
\hat{B} f(x) = (x) \left( -i\hbar \frac{d}{dx} \right) f(x) = \left( -i\hbar x \frac{d}{dx} \right) f(x) \\
\therefore \quad \hat{B} = -i\hbar x \frac{d}{dx} = \hat{x}\hat{p} \\
\therefore \quad \hat{x}\hat{p} \neq \hat{p}\hat{x}
\]

Define **commutator**

For two operators \( \hat{A} \) and \( \hat{B} \),

\[
\left[ \hat{A}, \hat{B} \right] = \hat{A}\hat{B} - \hat{B}\hat{A} = \hat{C}
\]

*Example:*

\[
\left[ \hat{x}, \hat{p} \right] = \hat{x}\hat{p} - \hat{p}\hat{x} = i\hbar \neq 0!
\]
Important general statements about commutators:

1) For operators that **commute**

\[
[A, B] = 0
\]

- it is possible to find a set of wavefunctions that are eigenfunctions of both operators simultaneously.

e.g. can find wavefunctions \( \psi_n \) such that

\[
\hat{A}\psi_n = a_n \psi_n \quad \text{and} \quad \hat{B}\psi_n = b_n \psi_n
\]

- This means that we can know the exact values of both observables \( A \) and \( B \) simultaneously (no uncertainty limitation).

2) For operators that do **not** commute

\[
[A, B] \neq 0
\]

- it is **not** possible to find a set of wavefunctions that are simultaneous eigenfunctions of both operators.

- This means that we **cannot** know the exact values of both observables \( A \) and \( B \) simultaneously \( \Rightarrow \) uncertainty!

\[
e.g. \quad \left[ \hat{x}, \hat{p} \right] = i\hbar \neq 0 \quad \Rightarrow \quad \Delta x \Delta p \geq \frac{\hbar}{2}
\]