I. Tunneling and Pictures (25 POINTS)

\[ V(x) = \begin{cases} \infty & |x| > a/2 \\ 0 & a/4 \leq |x| \leq a/2 \\ V_0 = \left( \frac{h^2}{8ma^2} \right) & |x| < a/4 \end{cases} \]

Regions I and V

Regions II and IV

Region III

The energy of the lowest level, \( E_n \), \( n = 1 \) is near \( E_1^{(0)} = \left( \frac{h^2}{8ma^2} \right) 9 \) and the second level, \( E_n \), \( n = 2 \), is near \( E_2^{(0)} = \left( \frac{h^2}{8ma^2} \right) 4 \).

A. (8 points) Sketch \( \psi_1(x) \) and \( \psi_2(x) \) on the figure above. In addition, specify below the qualitatively most important features that your sketch
of \( \psi_1(x) \) and \( \psi_2(x) \) must display inside Region III and at the borders of Region III.

B. (3 points) What do you know about \( \psi_1(0) \) and \( \frac{d\psi_1}{dx} \bigg|_{x=0} \) without solving for \( E_1 \) and \( \psi_1 \)?

(i) Is \( \psi_1(0) = 0? \)

No. \( \psi_1 \) cannot have a node and still be the lowest energy state.

(ii) Does \( \psi_1(0) \) have the same sign as \( \psi_1(a/2) \)?

Yes.

(iii) Is \( \frac{d\psi_1}{dx} \bigg|_{x=0} = 0? \)

Yes, because \( \psi_1 \) is a symmetric function.

C. (3 points) What do you know about \( \psi_2(0) \) and \( \frac{d\psi_2}{dx} \bigg|_{x=0} \) without solving for \( E_2 \) and \( \psi_2 \)?

(i) Is \( \psi_2(0) = 0? \)

Yes. \( \psi_2 \) must be antisymmetric and have a node at \( x = 0 \).

(ii) Is \( \frac{d\psi_2}{dx} \bigg|_{x=0} = 0? \)

No.

D. (3 points) In the table below, in the last column, place an X next to the mathematical form of \( \psi_1(x) \) in Region III.

<table>
<thead>
<tr>
<th></th>
<th>(i)</th>
<th>(ii)</th>
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<tbody>
<tr>
<td></td>
<td>( e^{kx} )</td>
<td>( e^{-kx} )</td>
<td>( \sin kx ) or ( \cos kx )</td>
<td>( e^{ikx} + e^{-ikx} )</td>
<td>( e^{ikx} - e^{-ikx} )</td>
<td>something else</td>
</tr>
</tbody>
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E. (3 points) Does the exact \( E_1 \) level lie above or below \( E_1^{(0)} \)?

Yes. \( \psi_1(x) \) feels the barrier strongly, which results in an increase in energy so that \( E_1 \gg E_1^{(0)} \).

F. (5 points) For the exact \( E_2 \) level, is the energy difference, \( |E_2 - E_2^{(0)}| \), larger or smaller than \( |E_1 - E_1^{(0)}| \)? Explain why.

The \( E_2 \) level hardly feels the barrier. It is shifted only slightly to higher energy than \( E_2^{(0)} \).

\[ |E_2 - E_2^{(0)}| \ll |E_1 - E_1^{(0)}| \]
II. Measurement Theory

Consider the Particle in an Infinite Box "superposition state" wavefunction,

\[ \psi_{1,2} = (1/3)^{1/2} \psi_1 + (2/3)^{1/2} \psi_2 \]

where \( E_1 \) is the eigen-energy of \( \psi_1 \) and \( E_2 \) is the eigen-energy of \( \psi_2 \).

A. (5 points) Suppose you do one experiment to measure the energy of \( \psi_{1,2} \)

Circle the possible result(s) of your measurement:

(i) \( E_1 \)  
(ii) \( E_2 \) \( \quad \) These are the eigenvalues of \( \psi_1 \) and \( \psi_2 \)
(iii) \( (1/3)E_1 + (2/3) E_2 \)
(iv) something else.

B. (5 points) Suppose you do 100 identical measurements to measure the energies of identical systems in state \( \psi_{1,2} \) What will you observe?

\[
\langle E \rangle = \frac{1}{3} E_1 + \frac{2}{3} E_2 \\
= \left[ \frac{h^2}{8ma^2} \right] \left[ \frac{1}{3} \cdot 1 + \frac{2}{3} \cdot 4 \right] \\
= \left[ \frac{h^2}{8ma^2} \right] \left[ \frac{1}{3} + \frac{8}{3} \right] \\
= \left[ \frac{h^2}{8ma^2} \right] [3]
\]

This value of \( \langle E \rangle \) is between \( E_1 \) and \( E_2 \) and is the weighted average energy.
III. Semiclassical Quantization  

Consider the two potential energy functions:

\[ V_1 \begin{cases} |x| \leq a/2, & V_1(x) = -|V_0| \\ |x| > a/2, & V_1(x) = 0 \end{cases} \]

\[ V_2 \begin{cases} |x| \leq a/4, & V_2(x) = -2|V_0| \\ |x| > a/4, & V_2(x) = 0 \end{cases} \]

A. (5 points) The semi-classical quantization equation below describes the number of levels below \( E \). Use this to compute the number of levels with energy less than 0 for \( V_1 \) and \( V_2 \).

\[
\left( \frac{2}{\hbar} \right) \int_{x_{E}}^{x_{E}} p_{E}(x)dx = n
\]

\[
p_{E} = \left[ 2m(E - V(x)) \right]^{1/2}
\]

For \( V_1 \)

\[
n = \left( \frac{2}{\hbar} \right) \int_{-a/2}^{a/2} \left[ 2m|V_0| \right]^{1/2} dx = \frac{\left[ 2m|V_0| \right]^{1/2} a}{\hbar}
\]

\[
p_{E} = \left[ 2m2|V_0| \right]^{1/2}
\]

For \( V_2 \)

\[
n = \left( \frac{2}{\hbar} \right) \int_{-a/4}^{a/4} \left[ 2m2|V_0| \right]^{1/2} dx = \frac{\left[ 4m|V_0| \right]^{1/2} a}{2\hbar}
\]

B. (5 points) \( V_1 \) and \( V_2 \) have the same product of width times depth, \( V_1 \) is \((a)|V_0|\) and \( V_2 \) is \((a/2)(2|V_0|)\), but \( V_1 \) and \( V_2 \) have different numbers of bound levels. Which has the larger fractional effect, increasing the depth of the potential by X% or increasing the width of the potential by X%?

\[ n(V_1) = 2^{1/2} n(V_2) \]

\( V_1 \), the wider well, has more levels than \( V_2 \), the deeper well. Width has larger fractional effect than depth.
IV. Creation/Annihilation Operators  

A. (2 points) Consider the integral

$$\int_{-\infty}^{\infty} \psi(x)^* a^\dagger a^\dagger a^\dagger a^\dagger a^\dagger a^\dagger a^\dagger a^\dagger a^\dagger a^\dagger a^\dagger a^\dagger a^\dagger a^\dagger a^\dagger a^\dagger a^\dagger a^\dagger \psi(x) \, dx.$$

For what values of $\nu - \nu'$ will the integral be non-zero (these are called selection rules)?

There are four $a^\dagger$ and three $a$, therefore $\nu = \nu' + 1$. The integral is non-zero when $\nu - \nu' = +1$.

B. (4 points) Let $\nu' = 4$ and $\nu$ be the value determined in part A to give a non-zero integral. Calculate the value of the above integral (DO NOT SIMPLIFY!).

Starting from right-most factor in the operator, with $\psi_{\nu=4}$ we have

$$[(5)(4)(3)(2)(2)(3)(4)]^{1/2}$$

 last first

C. (4 points)  Now consider the integral

$$\int_{-\infty}^{\infty} \psi(x)^* a^\dagger a^\dagger a^\dagger a^\dagger a^\dagger a^\dagger a^\dagger a^\dagger a^\dagger a^\dagger a^\dagger a^\dagger a^\dagger a^\dagger a^\dagger a^\dagger a^\dagger \psi(x) \, dx$$

Are the selection rules for $\nu' - \nu$ the same as in part A? Is the value of the non-zero integral for $\nu' = 4$ the same as in part B? If not, calculate the value of the integral (UNSIMPLIFIED!).

The $\nu - \nu'$ selection rule is the same for Part A but the numerical value of the integral is different. Starting from the right, we have

$$[(5)(5)(5)(5)(5)]^{1/2}$$

 last first

which is larger than the value in Part B.
D. (10 points) Derive the commutation rule $[\hat{N}, \hat{a}]$ starting from the definition of $\hat{N}.$

\[
\hat{N} = \hat{a} \hat{a}^\dagger \\
[\hat{a}, \hat{a}^\dagger] = +1 \\
[\hat{N}, \hat{a}^\dagger] = \hat{N} \hat{a} - \hat{a} \hat{N} = \hat{a}^\dagger \hat{a} \hat{a} - \hat{a} \hat{a}^\dagger \\
= \hat{a}^\dagger \hat{a}^2 - ([\hat{a}, \hat{a}^\dagger] + \hat{a}^\dagger \hat{a}) \hat{a} \\
= -([\hat{a}, \hat{a}^\dagger] \hat{a}) = -\hat{a} \\
[\hat{N}, \hat{a}] = -\hat{a}
\]

OR

\[
[\hat{N}, \hat{a}^\dagger] = \hat{N} \hat{a} - \hat{a} \hat{N} = [\hat{a}^\dagger, \hat{a}] \hat{a} + \hat{a} \hat{a}^\dagger \hat{a} - \hat{a} \hat{a}^\dagger \hat{a} \\
= [\hat{a}^\dagger, \hat{a}] \hat{a} = \hat{a} \\
[\hat{N}, \hat{a}] = -\hat{a}
\]
V. 〈x〉, 〈p〉, σ_x, σ_p and Time Evolution (35 POINTS)
of a Superposition State

\[ \hat{x} = \left[ \frac{\hbar}{2 \mu \omega} \right]^{1/2} (\hat{a}^\dagger + \hat{a}) \]
\[ \hat{p} = \left[ \frac{\hbar \mu}{2} \right]^{1/2} i (\hat{a}^\dagger - \hat{a}) \]

A. (5 points)  Show that \[ \hat{x}^2 = \left[ \frac{\hbar}{2 \mu \omega} \right] \left( \hat{a}^\dagger \hat{a}^2 + \hat{a}^\dagger \hat{a}^2 + 2 \hat{N} + 1 \right) \].

\[ \hat{x}^2 = \left[ \frac{\hbar}{2 \mu \omega} \right] \left( \hat{a}^\dagger \hat{a}^2 + \hat{a}^\dagger \hat{a}^2 + \hat{a}^\dagger \hat{a} + \hat{a}^\dagger \hat{a} \right) \]
\[ \hat{a}^\dagger \hat{a} = \hat{N} \]
\[ \hat{a}^\dagger \hat{a} = [\hat{a}, \hat{a}^\dagger] + \hat{a}^\dagger \hat{a} = 1 + \hat{N} \]
\[ \hat{x}^2 = \left[ \frac{\hbar}{2 \mu \omega} \right] \left( \hat{a}^\dagger \hat{a}^2 + \hat{a}^\dagger \hat{a}^2 + 2 \hat{N} + 1 \right) \]

B. (5 points)  Derive a similar expression for \[ \hat{p}^2 \]. (Be sure to combine \( \hat{a}^\dagger \hat{a} \) and \( \hat{a}^\dagger \hat{a} \) terms into an integer times \( \hat{N} \) plus another integer.

\[ \langle \hat{p}^2 \rangle = \left[ \frac{\hbar \mu \omega}{2} \right] (-1) \left( \hat{a}^\dagger \hat{a}^2 + \hat{a}^\dagger \hat{a}^2 - \hat{a}^\dagger \hat{a} - \hat{a}^\dagger \hat{a} \right) \]
\[ \hat{a}^\dagger \hat{a} = \hat{N} \]
\[ \hat{a}^\dagger \hat{a} = \hat{N} + 1 \]
\[ \langle \hat{p}^2 \rangle = \left[ \frac{\hbar \mu \omega}{2} \right] \left( \hat{a}^\dagger \hat{a}^2 + \hat{a}^\dagger \hat{a}^2 - 2 \hat{N} - 1 \right) \]
\[ = -\left[ \frac{\hbar \mu \omega}{2} \right] \left( \hat{a}^\dagger \hat{a}^2 + \hat{a}^\dagger \hat{a}^2 \right) + \left[ \frac{\hbar \mu \omega}{2} \right] (2 \hat{N} + 1) \]
C. (5 points) Evaluate $\sigma_x$ and $\sigma_p$. (Recall that $\sigma_x = \left[\langle \hat{x}^2 \rangle - \langle \hat{x} \rangle^2 \right]^{1/2}$).

$$\hat{x}^2 = \left[\frac{\hbar}{2\mu\omega}\right]^2 \left[\hat{a}^\dagger \hat{a} + \hat{\mathbf{a}}^2 + 2\mathbf{N} + 1\right]$$

we want $\int \psi^* \hat{x}^2 \psi \, dx$, selection rule is $\Delta \nu = 0$

$$\langle \hat{x}^2 \rangle = \left[\frac{\hbar}{2\mu\omega}\right]^2 (2\nu + 1)$$

we want $\int \psi^* \hat{x}^2 \psi \, dx$, selection rule is $\Delta \nu = 0$

$$\langle \hat{x} \rangle = 0 \text{ because selection rule is } \Delta \nu = \pm 1$$

$$\sigma_x = \left[\frac{\hbar}{2\mu\omega}\right]^{1/2} (2\nu + 1)^{1/2}$$

$$\hat{p}^2 = \left[\frac{\hbar \mu \omega}{2}\right] \left[-\hat{a}^\dagger \hat{a} - \hat{\mathbf{a}}^2 + 2\mathbf{N} + 1 \right]$$

$$\langle \hat{p}^2 \rangle = \left[\frac{\hbar \mu \omega}{2}\right] (2\nu + 1)$$

$$\sigma_p = \left[\frac{\hbar \mu \omega}{2}\right]^{1/2} (2\nu + 1)^{1/2}$$

Note that

$$\sigma_x \sigma_p = \left[\frac{\hbar}{2\mu\omega}\right]^{1/2} \left[\frac{\hbar \mu \omega}{2}\right]^{1/2} 2(\nu + 1/2)$$

$$= \frac{\hbar \mu \omega}{2}^{1/2} 2(\nu + 1/2)$$

$$= \hbar (\nu + 1/2)$$

D. (5 points) Show, using your results for $\hat{x}^2$ and $\hat{p}^2$, that

$$\hat{H} = \hat{p}^2 + \frac{k\hat{x}^2}{2\mu} = \hbar \omega [\mathbf{N} + 1/2].$$

(The contributions from $\hat{a}^2$ and $\hat{a}^\dagger$ exactly cancel.)

$$\frac{\hat{p}^2}{2\mu} = \frac{\hbar \mu \omega}{4\mu} \left[(2\nu + 1) - \hat{a}^2 - \hat{\mathbf{a}}^2 \right]$$

$$\frac{k\hat{x}^2}{2} = k \left[\frac{\hbar}{2\mu\omega}\right] \left[(2\nu + 1) + \hat{a}^2 + \hat{\mathbf{a}}^2 \right]$$

$$\frac{\hbar \mu \omega}{4\mu} = \frac{\hbar \omega}{4}$$

$$k \left[\frac{\hbar}{2\mu\omega}\right] = \frac{\hbar \omega}{4} \text{ because } \left(k / \mu \right) = \omega^2$$

$$\hat{H} = \hbar \omega [\mathbf{N} + 1/2]$$
E. (5 points) For $\Psi(x, t = 0) = c_0\psi_0 + c_1\psi_1 + c_2\psi_2$, write the time-dependent wavefunction, $\Psi(x,t)$.

$$
\Psi(x,t) = c_0 e^{-i0.5\omega t}\psi_0 + c_1 e^{-i1.5\omega t}\psi_1 + c_2 e^{-i2.5\omega t}\psi_2
$$

F. (5 points) Assume that $c_0$, $c_1$, and $c_2$ are real. Evaluate $\langle \hat{x}\rangle_t$ and show that $\langle x\rangle_t$ oscillates at angular frequency $\omega$. [HINT: $2\cos\theta = e^{i\theta} + e^{-i\theta}$]

We know the selection rule for $\hat{x}$ is $\Delta v = \pm 1$.

We know the selection rule for $\hat{x}^2$ is $\Delta v = 0, \pm 2$.

$$
\langle \hat{x}\rangle = \int \Psi \ast \hat{x}\Psi dx = |c_0|^2 0 + |c_1|^2 0 + |c_2|^2 0
$$

$$
+ c_0 c_1 x_{10} e^{-i\omega t} + c_1 c_0 x_{10} e^{i\omega t}
$$

$$
+ c_1 c_2 x_{12} e^{-i\omega t} + c_2 c_1 e^{i\omega t}
$$

$$
+ c_0 c_2 0 + c_2 c_0 0
$$

Thus $\langle \hat{x}\rangle = 2c_0 c_1 x_{01} \cos\omega t + 2c_1 c_2 x_{12} \cos\omega t$ because $x_{01} = x_{10}$ and $x_{12} = x_{21}$

$x_{21} = 2^{1/2} x_{10}$ could give additional simplification.

G. (5 points) Evaluate $\langle \hat{x}^2\rangle_t$. Show that $\langle \hat{x}^2\rangle_t$ includes a contribution that oscillates at an angular frequency of $2\omega$.

$$
\langle \hat{x}^2\rangle = \frac{\hbar}{2\mu\omega} \left[ a^\dagger a + a^2 \right] + \frac{2\hat{N} + 1}{\hbar}
$$

these give $\cos 2\omega t$

this gives a $t$-independent term
Some Possibly Useful Constants and Formulas

\( h = 6.63 \times 10^{-34} \text{ J} \cdot \text{s} \quad \hbar = 1.054 \times 10^{-34} \text{ J} \cdot \text{s} \)

\( \varepsilon_0 = 8.854 \times 10^{-12} \text{ C}^2 \text{ kg}^{-1} \text{ m}^{-3} \)

\( c = 3.00 \times 10^8 \text{ m/s} \quad c = \lambda \nu \quad \lambda = h/\rho \)

\( m_e = 9.11 \times 10^{-31} \text{ kg} \quad m_H = 1.67 \times 10^{-27} \text{ kg} \)

\( 1 \text{ eV} = 1.602 \times 10^{-19} \text{ J} \quad e = 1.602 \times 10^{-19} \text{ C} \)

\( E = h \nu \quad a_0 = 5.29 \times 10^{-11} \text{ m} \quad e^{\pm i\theta} = \cos \theta \pm i \sin \theta \)

\( \sqrt{\frac{1}{\lambda}} = R_H \left( \frac{1}{n_1^2} - \frac{1}{n_2^2} \right) \) \quad \text{where} \quad R_H = \frac{m e^4}{8 \varepsilon_0^2 \hbar^3 c} = 109,678 \text{ cm}^{-1} \)

Free particle:

\( E = \frac{\hbar^2 k^2}{2m} \quad \psi(x) = A \cos(kx) + B \sin(kx) \)

Particle in a box:

\( E_n = \frac{\hbar^2}{8ma^2} n^2 = E_1 n^2 \quad \psi(0 \leq x \leq a) = \left( \frac{2}{a} \right)^{1/2} \sin \left( \frac{n \pi x}{a} \right) \quad n = 1, 2, \ldots \)

Harmonic oscillator:

\( E_n = \left( n + \frac{1}{2} \right) \hbar \omega \quad \text{[units of } \omega \text{ are radians/s]} \)

\( \psi_0 (x) = \left( \frac{\alpha}{\pi} \right)^{1/4} e^{-\alpha x^2/2} \quad \psi_1 (x) = \frac{1}{\sqrt{2}} \left( \frac{\alpha}{\pi} \right)^{1/4} \left( 2 \alpha^{1/2} x \right) e^{-\alpha x^2/2} \quad \psi_2 (x) = \frac{1}{\sqrt{8}} \left( \frac{\alpha}{\pi} \right)^{1/4} \left( 4 \alpha x^2 - 2 \right) e^{-\alpha x^2/2} \)

\( \hat{x} = \sqrt{\frac{m \omega}{\hbar}} \hat{x} \quad \hat{p} = \sqrt{\frac{1}{\hbar m \omega}} \hat{p} \quad \text{[units of } \omega \text{ are radians/s]} \)

\( a = \frac{1}{\sqrt{2}} \left( \hat{x} + i \hat{p} \right) \quad \frac{\hat{H}}{\hbar \omega} = aa^\dagger - \frac{1}{2} = a^\dagger a + \frac{1}{2} \quad \hat{N} = a^\dagger a \)

\( a^\dagger = \frac{1}{\sqrt{2}} \left( \hat{x} - i \hat{p} \right) \)

\( 2 \pi c \bar{\omega} = \omega \quad \text{[units of } \bar{\omega} \text{ are cm}^{-1}] \)
Semi-Classical

\[ \lambda = \frac{h}{p} \]

\[ p_{\text{classical}}(x) = \left[ 2m(E - V(x)) \right]^{1/2} \]

period: \[ \tau = \frac{1}{\nu} = \frac{2\pi}{\omega} \]

For a thin barrier of width \( \varepsilon \) where \( \varepsilon \) is very small, located at \( x_0 \), and height \( V(x_0) \):

\[ H_{nn}^{(1)} = \int_{x_0 - \varepsilon/2}^{x_0 + \varepsilon/2} \psi_n^{(0)*} V(x) \psi_n^{(0)} dx = \varepsilon V(x_0) \left| \psi_n^{(0)}(x_0) \right|^2 \]