Slowly Applied (Adiabatic) Perturbation

All of our perturbations so far have been applied suddenly at \( t > t_0 \) (step function)

\[
V(t) = \theta(t - t_0) V(t)
\]

This leads to unphysical consequences—you generally can’t turn on a perturbation fast enough to appear instantaneous. Since first-order P.T. says that the transition amplitude is related to the Fourier Transform of the perturbation, this leads to additional Fourier components in the spectral dependence of the perturbation—even for a monochromatic perturbation!

So, let’s apply a perturbation slowly . . .

\[
V(t) = V e^{\eta t}
\]

\( \eta \) small and positive

The system is prepared in state \( |\ell\rangle \) at \( t = -\infty \). Find \( P_k(t) \).

\[
b_k = \langle k | U | \ell \rangle = \frac{-i}{\hbar} \int_{-\infty}^{t} d\tau e^{i\omega_k \tau} \langle k | V | \ell \rangle e^{\eta \tau}
\]

\[
b_k = \frac{-iV_{k\ell}}{\hbar} \frac{\exp[\eta t + i\omega_{k\ell} t]}{\eta + i\omega_{k\ell}}
\]

\[
= V_{k\ell} \frac{\exp[\eta t + i(E_k - E_\ell) t / \hbar]}{E_k - E_\ell + i\eta \hbar}
\]

\[
P_k = |b_k|^2 = \frac{|V_{k\ell}|^2 \exp[2\eta t]}{\eta^2 + \omega_{k\ell}^2} = \frac{|V_{k\ell}|^2 \exp[2\eta t]}{(E_k - E_\ell)^2 + (\eta \hbar)^2}
\]
This is a Lorentzian lineshape in $\omega_k\ell$ with width $2\eta\hbar$.

**Gradually Applied Perturbation**

$2\eta\hbar$

**Step Response Perturbation**

$2\pi\hbar/t$

The gradually turned on perturbation has a width dependent on the turn-on rate, and is independent of time. (The amplitude grows exponentially in time.) Notice, there are no nodes in $P_k$.

$\eta^{-1}$ is the effective turn-on time of the perturbation:

Now, let’s calculate the transition rate:

$$w_{k\ell} = \frac{\partial P_k}{\partial t} = \frac{|V_{k\ell}|^2}{\hbar^2} \frac{2\eta e^{2\eta t}}{\eta^2 + \omega_{k\ell}^2}$$

Look at the adiabatic limit: $\eta \to 0$.

setting $e^{2\eta t} \to 1$; and using $\lim_{\eta \to 0} \frac{\eta}{\eta^2 + \omega_{k\ell}^2} = \pi \delta(\omega_{k\ell})$

$$w_{k\ell} = \frac{2\pi}{\hbar^2} |V_{k\ell}|^2 \delta(\omega_{k\ell}) = \frac{2\pi}{\hbar} |V_{k\ell}|^2 \delta(E_k - E_{\ell})$$

We get Fermi’s Golden Rule—indepedent of how perturbation is introduced!
If we gradually apply the Harmonic Perturbation,

\[ V(t) = V e^{\eta t} \cos \omega t \]

\[ b_k = \frac{-i}{\hbar} \int_{-\infty}^{t} d\tau V_{kl} e^{i\omega t + \eta \tau} \left[ \frac{e^{i\omega \tau} + e^{-i\omega \tau}}{2} \right] \]

\[ = \frac{V_{kl} \omega \eta}{2\hbar} e^{-i\omega \tau} \left[ \frac{e^{i(\omega + \omega) \tau}}{-(\omega + \omega) + i\eta} + \frac{e^{i(\omega - \omega) \tau}}{-(\omega - \omega) + i\eta} \right] \]

Again, we have a resonant and anti-resonant term, which are now broadened by \( \eta \).

If we only consider absorption:

\[ P_k = |b_k|^2 = \frac{|V_{kl}|^2}{4\hbar^2} e^{2\eta \omega} \frac{1}{(\omega_{kl} - \omega)^2 + \eta^2} \]

which is the Lorentzian lineshape centered at \( \omega_{kl} = \omega \) with width \( \Delta \omega = 2\eta \).

Again, we can calculate the adiabatic limit, setting \( \eta \to 0 \). We will calculate the rate of transitions \( \omega_{kl} = \partial P_k / \partial t \). But let’s restrict ourselves to long enough times that the harmonic perturbation has cycled a few times (this allows us to neglect cross terms) \( \to \) resonances sharpen.

\[ w_{kl} = \frac{\pi}{2\hbar^2} |V_{kl}|^2 \left[ \delta(\omega_{kl} - \omega) + \delta(\omega_{kl} + \omega) \right] \]