Brief Notes #1
Events and Their Probability

• Definitions

Experiment: a set of conditions under which some variable is observed
Outcome of an experiment: the result of the observation (a sample point)
Sample Space, S: collection of all possible outcomes (sample points) of an experiment
Event: a collection of sample points

• Operations with events

1. Complementation
   \[ A^c \]

2. Intersection
   \[ A \cap B \]

3. Union
   \[ A \cup B \]

• Properties of events

1. Mutual Exclusiveness - intersection of events is the null set (\(A_i \cap A_j = \emptyset\), for all \(i \neq j\))
2. Collective Exhaustiveness (C.E.) - union of events is sample space (\(A_1 \cup A_2 \cup \ldots \cup A_n = S\))
3. If the events \(\{A_1, A_2, \ldots, A_n\}\) are both mutually exclusive and collectively exhaustive, they form a partition of the sample space, S.

• Probability of events

• Relative frequency \(f_E\) and limit of relative frequency \(F_E\) of an event E
   \[ f_E = \frac{n_E}{n} \]
   \[ F_E = \lim_{n \to \infty} f_E = \lim_{n \to \infty} \frac{n_E}{n} \]

• Properties of relative frequency (the same is true for the limit of relative frequency)
1. $0 \leq f_E \leq 1$
2. $f_S = 1$
3. $f_{(A \cup B)} = f_A + f_B$ if $A$ and $B$ are mutually exclusive

**Properties/axioms of probability**

1. $0 \leq P(A) \leq 1$
2. $P(S) = 1$
3. $P(A \cup B) = P(A) + P(B)$ if $A$ and $B$ are mutually exclusive

**Two consequences of the axioms of probability theory**

1. $P(A^c) = 1 - P(A)$
2. $P(A \cup B) = P(A) + P(B) - P(A \cap B)$, for any two events $A$ and $B$,
   \[ \Rightarrow P(A \cap B) = P(A) + P(B) - P(A \cup B) \]

**Conditional Probability**

Definition:

\[
P(A \mid B) = \frac{P(A \cap B)}{P(B)}
\]

Therefore, $P(A \cap B)$ can also be obtained as

\[
P(A \cap B) = P(B)P(A\mid B) = P(A)P(B\mid A)
\]

**Total Probability Theorem**

Let \{B_1, B_2, ..., B_n\} be a set of mutually exclusive and collectively exhaustive events and let $A$ be any other event. Then the marginal probability of $A$ can be obtained as:

\[
P(A) = \sum_i P(A \cap B_i) = \sum_i P(B_i)P(A \mid B_i)
\]

**Independent events**

$A$ and $B$ are independent if:

- $P(A \mid B) = P(A)$, or equivalently if
- $P(B \mid A) = P(B)$, or if
- $P(A \cap B) = P(A) P(B)$

**Bayes' Theorem**
\[ P(A \mid B) = P(A) \frac{P(B \mid A)}{P(B)} \]

Using Total Probability Theorem, \( P(B) \) can be expressed in terms of \( P(A), P(A^c) = 1 - P(A) \), and the conditional probabilities \( P(B \mid A) \) and \( P(B \mid A^c) \):

\[ P(B) = P(A)P(B \mid A) + P(A^c)P(B \mid A^c) \]

So Bayes’ Theorem can be rewritten as:

\[ P(A \mid B) = P(A) \frac{P(B \mid A)}{P(A)P(B \mid A) + P(A^c)P(B \mid A^c)} \]