Introduction to Network Models

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Lecture 11
Small-world phenomenon

- Stanley Milgram's experiment ⇒ six degrees of separation

- Get letter from ‘starter’ to target by forwarding to acquaintances
  ⇒ Letters arrive with a median of six steps

- Two rather surprising facts
  ⇒ 1) Short paths between two nodes exist in abundance
  ⇒ 2) People without global knowledge can find these paths

Milgram, Stanley. "The Small-World Today." *Psychology Today* 1 (1967): 61–67. © Sussex Publishers. All rights reserved. This content is excluded from our Creative Commons license. For more information, see [https://ocw.mit.edu/help/faq-fair-use/](https://ocw.mit.edu/help/faq-fair-use/).
How can this happen?

- Pure exponential growth produces a small world

- Triadic closure reduces the growth rate

- Can a model exhibit both many closed triads and very short paths?
Clustering coefficient

- **Q:** What fraction of \( v \)'s neighbors are themselves connected?

- The **clustering coefficient** \( \text{cl}(v) \) of \( v \in V \) is

\[
\text{cl}(v) = \frac{2|E_v|}{d_v(d_v - 1)} \in [0, 1]
\]

\( \Rightarrow |E_v| \) is the number of edges among \( v \)'s neighbors

- An indication of the extent to which edges ‘cluster’

- The global (average) clustering coefficient is

\[
\text{cl}(G) = \frac{1}{|V|} \sum_{v \in V} \text{cl}(v)
\]
Do we really need another model for this?

<table>
<thead>
<tr>
<th>Network</th>
<th>size</th>
<th>av. shortest path</th>
<th>Shortest path in fitted random graph</th>
<th>Clustering (averaged over vertices)</th>
<th>Clustering in random graph</th>
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</thead>
<tbody>
<tr>
<td>Film actors</td>
<td>225,226</td>
<td>3.65</td>
<td>2.99</td>
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<td>MEDLINE co-authorship</td>
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<td>1.8 x 10^-4</td>
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<td>3.04</td>
<td>0.32</td>
<td>0.026</td>
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<tr>
<td>C.Elegans</td>
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<td>2.25</td>
<td>0.28</td>
<td>0.05</td>
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</tbody>
</table>
Watts-Strogatz small-world model

- Reconciling short paths and high clustering coefficients
- Desired number of nodes $n$, average degree $K$ and probability $p$
  - 1) Construct a circle of $n$ nodes
  - 2) Connect each node to its $K$ closest neighbors
  - 3) With probability $p$ rewire each edge uniformly

Small $p$: regular lattice. Large $p$: close to ER graph
- There is a sweet spot in between

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Watts, Duncan J., and Steven H. Strogatz. "Collective dynamics of 'small-world' networks." *Nature* 393 (1998): 441–42. © Springer Nature. All rights reserved. This content is excluded from our Creative Commons license. For more information, see [https://ocw.mit.edu/help/faq-fair-use/](https://ocw.mit.edu/help/faq-fair-use/).
Intermediate rewiring probabilities

- Plot clustering coefficient and average shortest path
  
  ⇒ As a function of the rewiring probability $p$

- What happens for $p \approx 0.01$?


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