1.022 Introduction to Network Models

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Lecture 9
Behavior of $G_{n,p}$ for increasing $p$

- ER graphs exhibit phase transitions
  - Sharp transitions between behaviors as $n \to \infty$

**ER connectivity theorem**

- A threshold function for the connectivity of $G_{n,p(n)}$ is $p(n) = \frac{\ln(n)}{n}$
- Let $p(n) = \lambda \frac{\ln(n)}{n}$ then
  - If $\lambda < 1$ \( \Rightarrow \) $\mathbb{P}(\text{connected}) \to 0$ as $n \to \infty$
  - If $\lambda > 1$ \( \Rightarrow \) $\mathbb{P}(\text{connected}) \to 1$ as $n \to \infty$
To show disconnectedness, it is sufficient to show that the probability that there exists at least one isolated node goes to 1.

- Let $I_i$ be a Bernoulli random variable defined as
  \[
  I_i = \begin{cases} 
  1 & \text{if node } i \text{ is isolated}, \\
  0 & \text{otherwise}.
  \end{cases}
  \]

- We can write the probability that an individual node is isolated as
  \[
  q = \mathbb{P}(I_i = 1) = (1 - p)^{n-1} \approx e^{-pn} = e^{-\lambda \log(n)} = n^{-\lambda},
  \]
  where we use \( \lim_{n \to \infty} \left(1 - \frac{a}{n}\right)^{n} = e^{-a} \) to get the approximation.

- Let $X = \sum_{i=1}^{n} I_i$ denote the total number of isolated nodes. Then, we have
  \[
  \mathbb{E}[X] = n \cdot n^{-\lambda}.
  \]
Sketch of the Proof

- For $\lambda < 1$, we have $\mathbb{E}[X] \to \infty$. We want to show that this implies $\mathbb{P}(X = 0) \to 0$.

- In general, this is not true. But, here it holds.

- We can show that the variance of $X$ is of the same order as its mean:

$$\text{var}(X) \sim \mathbb{E}[X],$$

where $a(n) \sim b(n)$ denotes $\frac{a(n)}{b(n)} \to 1$ as $n \to \infty$.

- This implies that

$$\mathbb{E}[X] \sim \text{var}(X) \geq (0 - \mathbb{E}[X])^2 \mathbb{P}(X = 0),$$

and therefore,

$$\mathbb{P}(X = 0) \leq \frac{\mathbb{E}[X]}{\mathbb{E}[X]^2} = \frac{1}{\mathbb{E}[X]} \to 0.$$

- It follows that $\mathbb{P}(\text{at least one isolated node}) \to 1$ and therefore, $\mathbb{P}(\text{disconnected}) \to 1$ as $n \to \infty$, completing the proof.
If \( p(n) = \lambda \frac{\log(n)}{n} \) with \( \lambda > 1 \), then \( \mathbb{P}(\text{disconnected}) \to 0 \).

- \( \mathbb{E}[X] = n^{1-\lambda} \to 0 \) for \( \lambda > 1 \). Almost surely no isolated node.
- We need more to establish connectivity.
- The event “graph is disconnected” is equivalent to the existence of \( k \) nodes without an edge to the remaining nodes, for some \( k \leq n/2 \).
- We have

\[
\mathbb{P}(\{1, \ldots, k\} \text{ not connected to the rest}) = (1 - p)^{k(n-k)} \to
\]

\[
\mathbb{P}(\exists k \text{ nodes not connected to the rest}) = \binom{n}{k} (1 - p)^{k(n-k)} \to
\]

\[
\mathbb{P}(\text{disconnected graph}) \leq \sum_{k=1}^{n/2} \binom{n}{k} (1 - p)^{k(n-k)}.
\]

- bounding RHS and some algebraic manipulation yields

\[
\mathbb{P}(\text{disconnected graph}) \leq C n^{-\frac{1}{1+\lambda}} \to 0.
\]
**Figure:** Emergence of connectedness: a random network on 50 nodes with $p = 0.10$. 
ER giant component theorem

- A threshold function for the emergence of a giant component in $G_{n,p(n)}$ is $p(n) = \frac{1}{n}$

- Let $p(n) = \frac{\lambda}{n}$ then
  - If $\lambda < 1$  $\Rightarrow$ Size of largest component $\sim \ln(n)$ as $n \to \infty$
  - If $\lambda > 1$  $\Rightarrow$ Size of largest component $\sim n$ as $n \to \infty$

- In fact, the size of giant component satisfies
  
  \[ 1 - q = e^{-\lambda q} \]
q: giant component size, the probability of a randomly chosen node is in giant component

Consider a vertex not in the giant component: For every other vertex $j$ either

- $i$ is not connected to $j$ by an edge, or
- $i$ is connected to $j$ but $j$ is not in the giant component.

This gives

$$1 - q = (1 - p + p(1 - q))^{n-1} = (1 - pq)^{n-1},$$

RHS can be approximated as $e^{-p(n-1)q} \sim e^{-\lambda q}$ when $n \to \infty$. 
- \( q = 0 \) is always a solution of \( 1 - q = e^{-\lambda q} \).

- Looking at the derivative of both sides at \( q = 0 \), we can show the existence of a nonzero solution if and only if \( \lambda > 1 \).