Mechanics of Material Systems
(Mechanics and Durability of Solids I)

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Lecture: MWF 1 // Recitation: F 3:00-4:30
Part III: Elasticity and Elasticity Bounds

5. Thermoelasticity
Part I. Deformation and Strain
  1 Description of Finite Deformation
  2 Infinitesimal Deformation

Part II. Momentum Balance and Stresses
  3 Momentum Balance
  4 Stress States / Failure Criterion

Part III. Elasticity and Elasticity Bounds
  5 Thermoelasticity,
  6 Variational Methods

Part IV. Plasticity and Yield Design
  7 1D-Plasticity – An Energy Approach
  8 Plasticity Models
  9 Limit Analysis and Yield Design
### The Necessity of Material Laws

<table>
<thead>
<tr>
<th>UNKNOWNS</th>
<th>EQUATIONS</th>
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</thead>
<tbody>
<tr>
<td>- 6 strains $\varepsilon_{ij} = \varepsilon_{ji}$</td>
<td>- 3 Momentum Balance $\sigma_{ij,j} + \rho f_i = 0$</td>
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<tr>
<td>- 6 stresses $\sigma_{ij} = \sigma_{ji}$</td>
<td>- 6 Strain-Displacement Relations: $2\varepsilon_{ij} = \xi_{i,j} + \xi_{j,i}$</td>
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<tr>
<td>- 3 displacements $\xi_i$</td>
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### Equations

- Momentum Balance: $\sigma_{ij,j} + \rho f_i = 0$
- Strain-Displacement Relations: $2\varepsilon_{ij} = \xi_{i,j} + \xi_{j,i}$

### Unknowns

- $\Sigma = 15$
- $\Delta = 15 - 9 = 6$

6 Missing Relations

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1D Think Model of Elasticity

Unique relation

\[ \sigma \leftrightarrow \varepsilon; \quad \sigma = \sigma(\varepsilon); \quad \varepsilon = \varepsilon(\sigma) \]

Nonlinear Elastic Behavior

Tangent Modulus

Linear Elastic Behavior

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Elasticity Potential

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\( \epsilon [L] \)

\( \sigma \)

\( 1 [L] \)

\( \psi = \frac{1}{2} E \epsilon^2 \)

\( \psi = \psi (\epsilon_{ij}) \)

\( \sigma = \frac{\partial \psi}{\partial \epsilon} \)

\( \sigma_{ij} = \frac{\partial \psi}{\partial \epsilon_{ij}} \)

\( \varphi dt = \sigma d\epsilon - d\psi = 0 \)

Work = Helmholtz Energy

\( \psi \) = Stored Energy

\( 1^{st} + 2^{nd} \) Law:

\( \psi \) = Helmholtz Energy

1D-Model:

3D-Model:
Hydrostatic Test

\[ \sigma_m = -p = K \frac{\Delta V}{V} \]

Bulk Modulus

Triaxial Test

\[ -\sigma_{II} = -\sigma_{III} \]

\[ \sigma_I - \sigma_{III} = 2G (\varepsilon_I - \varepsilon_{III}) \]

Shear Modulus

Isotropic Elastic Material Properties
Shear Modulus – Triaxial Test

Mohr Representation

\[
\frac{(\sigma_I - \sigma_{III})}{2}
\]

\[
\frac{(\varepsilon_I - \varepsilon_{III})}{2}
\]

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Boussinesq Problem
Direct Solving Methods in Elasticity
Displacement Method

Start $\xi$

$2\varepsilon_{ij} = \xi_{i,j} + \xi_{j,i}$

Linear Isotropic Elasticity

$\sigma_{ij} = \sigma_{ij}^0 + 2\mu \varepsilon_{ij} + \lambda \varepsilon_{kk} \delta_{ij}$

Momentum Balance

$\sigma_{ij,i} + \rho f_i = 0$

Integration

Solution

Displacement + Stress

Boundary Condition

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Exercise: Soil layer under uniform surface pressure

Application of Displacement Method

\[ \xi = u(X) \mathbf{e}_x \]

\[ T^d(0) = p \mathbf{e}_x \]

\[ u^d(H) = 0 \]
Training Set: Cylinder Tube … Deep Tunnel

Illustration of cylinder strain components

\[ l' = l(1 + \varepsilon_{\theta\theta}) \]

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Vessel Formula Revisited
(Elasticity)

Maximum Shear in Thick Cylinder Tube

Mohr Representation
Theorem of Superposition
Applied to Deep Tunneling in Elastic Soil/Rock

\[ \sigma_0 = -p_0 \mathbf{1} \]

Natural Isotropic Prestress

Excavation = Removal of ‘Support’-Stress

Thick tube solution for outer radius \( R_1 \to \infty \)

\[ r = R_0: T^d = -p_0 \mathbf{e}_r \]

Linear Elastic Solution

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\[ T^d = -p_0 \mathbf{e}_r \]

\[ r = R_0: \]