Probabilistic Planning 2

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3-27-2004
Topics

- PERT (Cont’d)
  - Review
  - Merge node bias
  - PNet refinement

- Monte Carlo

- Simulation approaches
  - General
  - Demo
  - Process Interaction
  - Activity Scanning
**PERT Basics**

- Expresses uncertainty in *activity* duration
  - Beta distribution assumed for activities

- Assume normally distributed *project* duration
  - Project Duration Tends to be Normally Distributed (approx. sum of random variables)
  - Assumes Independent Activity Durations - Not Always Satisfied
Stochastic Approach

- Optimistic
- *Most Likely* (*mode – not mean*)
- Pessimistic
- Expected Duration
- Variance
- Standard Deviation

\[
\bar{d} = \frac{1}{3} \left[ 2m + \frac{1}{2} (a+b) \right] = \frac{a + 4m + b}{6}
\]

\[
\nu = s^2
\]

\[
s = \frac{b - a}{6}
\]
Recall: Steps in PERT Analysis

- For each activity $k$
  - Obtain $a_k$, $m_k$ (mode) and $b_k$
  - Compute expected activity duration (mean) $d_k = t_e$
  - Compute activity variance $v_k = s^2$

- Compute expected project duration $D = T_e$ using standard CPM algorithm

- Compute Project Variance $V = S^2$ as sum of critical path activity variance (*this assumes independence*)
  - In case of multiple critical paths use the one with the largest variance

- Compute probability complete project by time $t$
  - Assuming project duration normally distributed
## PERT Example

<table>
<thead>
<tr>
<th>Activity</th>
<th>Predecessor</th>
<th>a</th>
<th>m</th>
<th>b</th>
<th>d</th>
<th>v</th>
</tr>
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<tbody>
<tr>
<td>A</td>
<td>-</td>
<td>1</td>
<td>2</td>
<td>4</td>
<td>2.17</td>
<td>0.25</td>
</tr>
<tr>
<td>B</td>
<td>-</td>
<td>5</td>
<td>6</td>
<td>7</td>
<td>6.00</td>
<td>0.11</td>
</tr>
<tr>
<td>C</td>
<td>-</td>
<td>2</td>
<td>4</td>
<td>5</td>
<td>3.83</td>
<td>0.25</td>
</tr>
<tr>
<td>D</td>
<td>A</td>
<td>1</td>
<td>3</td>
<td>4</td>
<td>2.83</td>
<td>0.25</td>
</tr>
<tr>
<td>E</td>
<td>C</td>
<td>4</td>
<td>5</td>
<td>7</td>
<td>5.17</td>
<td>0.25</td>
</tr>
<tr>
<td>F</td>
<td>A</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td>4.00</td>
<td>0.11</td>
</tr>
<tr>
<td>G</td>
<td>B,D,E</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>2.00</td>
<td>0.11</td>
</tr>
</tbody>
</table>
Activity on Node Example
Forward Pass
Backward Pass
\[ T_e = 11 \]

\[ = 0.25 + 0.25 + 0.11111 \]
\[ = 0.61111 \]

\[ S = \sqrt{0.61111} \]
\[ = 0.7817 \]
PERT Analysis-Probability of Ending before 10 (Critical Path Only)

\[
P(T \leq T_d) = P(T \leq 10) \\
= P\left(z \leq \frac{10 - T_e}{S}\right) \\
= P\left(z \leq \frac{10 - 11}{0.7817}\right) \\
= P(z \leq -1.2793) \\
= 1 - P(z \leq 1.2793) \\
= 1 - 0.8997 \\
= 0.1003 \\
= 10\%
\]
PERT Analysis - Probability of Ending before 13
(Critical Path Only)

\[ P(T \leq 13) = P\left( z \leq \frac{13-11}{0.7817} \right) \]

\[ = P(z \leq 2.5585) \]

\[ = 0.9948 \]
PERT Analysis - Probability of Ending between 9 and 11.5 (CP Only)

\[ P(T_L \leq T \leq T_U) = P(9 < T \leq 11.5) \]

\[ = P(T \leq 11.5) - P(T \leq 9) \]

\[ = P\left(z \leq \frac{11.5 - 11}{0.7817}\right) - P\left(z \leq \frac{9 - 11}{0.7817}\right) \]

\[ = P(z \leq 0.6396) - P(z \leq -2.5585) \]

\[ = P(z \leq 0.6396) - [1 - P(z \leq 2.5585)] \]

\[ = 0.7389 - [1 - 0.9948] \]

\[ = 0.7389 - 0.0052 \]

\[ = 0.7337 \]
Topics

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  - Merge node bias
  - PNet refinement

- Monte Carlo

- Simulation approaches
  - General
  - Demo
  - Process Interaction
  - Activity Scanning
Merge Node Bias

- Misleading to consider only \textit{variance} from single predecessor for each node on critical path
  - Early start of node depends on \textit{maximum} of finish (or start) times of predecessors – including non-critical!
- Basically ES = RV that is max of (non-iid) RVs
- Effect stronger if have
  - More predecessors
  - Predecessors with almost equal timing
  - Less dependency among predecessors
- Consequence: \textit{Unrealistic optimism} with respect to expected completion times, but especially \textit{variance}
Example Merge Node

ES(C) = \text{Max}(EF(A), EF(B))

\mu = 10.777
\sigma = 1.55

Late Finish: N(10,1)
Late Finish: N(9,3)
Illustration of the “conventional” PERT statistical approach to the network
**Derived Parameters**

<table>
<thead>
<tr>
<th>ACTIVITY</th>
<th>TIME ESTIMATES</th>
<th>PATH 0-3-7-8 (Critical Path)</th>
<th>PATH 0-3-4-5-8</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>a</td>
<td>m</td>
<td>b</td>
</tr>
<tr>
<td>0-3</td>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>3-7</td>
<td>6</td>
<td>8</td>
<td>10</td>
</tr>
<tr>
<td>7-8</td>
<td>3.5</td>
<td>5</td>
<td>6.5</td>
</tr>
<tr>
<td>3-4</td>
<td>1</td>
<td>4</td>
<td>13</td>
</tr>
<tr>
<td>4-5</td>
<td>2</td>
<td>4</td>
<td>6</td>
</tr>
<tr>
<td>5-8</td>
<td>2</td>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td><strong>TOTALS</strong>*</td>
<td><strong>15.0</strong></td>
<td><strong>2.83</strong></td>
<td><strong>14.0</strong></td>
</tr>
</tbody>
</table>

**STANDARD DEVIATION** - 1.68 - 4.05

*The mean and variance of the duration of a path is merely the sum of the means and variances of the activities along the path in question; the standard deviation of the path duration is then obtained as the square root of its variance.*
Impact of Multiple Paths

Log Scale

(most critical times of both paths)

Critical Path
0-3-7-8

Near Critical Path
0-3-4-5-8

Both Paths

Both

Project scheduled duration time, $T_s$

Cumulative probability of path and project completion on or before time $T_s$

Path | Mean | Standard Deviation
--- | ---: | ---: |
0-3-7-8 | 15.0 | 1.68 |
0-3-4-5-8 | 14.0 | 4.05 |
0-3-4-5-8 | 16.3 | 2.38 |
Naïve Approach

- Consider variance from all paths entering a merge node

- Assume Probability $EF(i) < T = \prod_{j \in \text{Paths To}(i)} P(\text{EF}(j) < T)$
Recall
\[ T_e = 7 \]


\[ = 0.25 + 0.25 + 0.11 \]

\[ = 0.6111 \]

\[ S = \sqrt{0.6111} \]

\[ = 0.7817 \]
$P(T \leq 10) = P\left(z \leq \frac{10 - 7}{0.7817}\right)$

$= P(z \leq 3.8378)$

$= 0.99999$
$T_e = 8$

$S^2 = V[B] + V[G]$

$= 0.1111 + 0.1111$

$= 0.2222$

$S = \sqrt{0.2222}$

$= 0.4714$
PERT Analysis - BG Path Probability of Ending before 10

\[ P(T \leq 10) = P\left( z \leq \frac{10 - 8}{0.4714} \right) \]

\[ = P(z \leq 4.2429) \]

\[ = 0.99999 \]
PERT Analysis - ADG, BG and CEG Paths
Probability of Ending before 10

\[ P_c(T \leq 10) = P(T_{CEG} \leq 10)P(T_{ADG} \leq 10)P(T_{BG} \leq 10) \]

\[ = (0.1003)(0.9999)(0.9999) \]

\[ = 0.1003 \]

\[ = 10\% \]
PERT (cont):

- For the G finish within 10 days, all 3 paths must finish in 10 days or less (i.e. ADG and CEG and BG)
- Calculated as:
  \[ P(T \leq 10) = P(ADG \leq 10) \times P(CEG \leq 10) \times P(BG \leq 10) \]
- What is wrong with this equation?
- The equation assumes the path durations are independent!
- This cannot be if there are shared activities between the paths.
Example of Multiple Paths – Dependent and Independent

Activities with duration 2 have $\sigma=.707$
Activities with duration 4 have $\sigma=1.414$
PERT (cont):

- **A Solution: Use either**
  - PNet
  - Monte Carlo simulation
PNet

- Aims at addressing merge node bias
- Basically works by
  - Enumerate all paths $P$ s.t. $\text{Dur}(P) > \alpha \text{Dur}(\text{crit path})$
  - Rank paths by decreasing duration (by decreasing naively-estimated variance for ties)
  - Compute linear correlation coefficient between paths
  - Enter paths, eliminating any path whose correlation coefficient with a previously-entered path is > .5

$$P(T \leq \alpha) = \prod_{i=1}^{\# \text{remaining paths}} P(p_i \leq T)$$
PERT Disadvantages

- Validity of Beta distribution for activity durations
- Validity of central limit theorem for project duration
  - Activity durations are not independent!
- Take into consideration only critical path
  - Not just sum of random variables -- have max. at joins
  - Leads to overoptimism & underestimation of duration
- Multiple time estimates required to calibrate
  - Can be time consuming
Topics

- PERT (Cont’d)
  - Review
  - Merge node bias
  - PNet refinement
- Monte Carlo
- Simulation approaches
  - General
  - Demo
  - Process Interaction
  - Activity Scanning
Monte Carlo Simulation

Characteristics

- Replaces analytic solution with raw computing power
  - Avoids need to simplify to get analytic solution
  - No need to assume functional form of activity/project distributions
- Used by Van Slyke (1963)
- Aimed at solving the merge bias problem in PERT
- Allows determining the criticality index of an activity (Proportion of runs in which the activity was in the critical path)
- Hundreds to thousands of simulations needed
Monte Carlo Simulation Process

- Set the duration distribution for each activity
  - No functional form of distribution assumed
  - Could be joint distribution for multiple activities

- Iterate: for each “trial” (“realization”)
  - Sample random duration from each distributions
  - Find critical path & durations with standard CPM
    - Record these results

- Report recorded results
  - Duration distribution
  - Per-node criticality index (% runs where critical)
Network
### Monte Carlo Simulation Example

Statistics for Example Activities

<table>
<thead>
<tr>
<th>Activity</th>
<th>Optimistic Time, (a)</th>
<th>Most Likely Time, (m)</th>
<th>Pessimistic Time, (b)</th>
<th>Expected Value, (d)</th>
<th>Standard Deviation, (s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>2</td>
<td>5</td>
<td>8</td>
<td>5</td>
<td>1</td>
</tr>
<tr>
<td>B</td>
<td>1</td>
<td>3</td>
<td>5</td>
<td>3</td>
<td>0.66</td>
</tr>
<tr>
<td>C</td>
<td>7</td>
<td>8</td>
<td>9</td>
<td>8</td>
<td>0.33</td>
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<tr>
<td>D</td>
<td>4</td>
<td>7</td>
<td>10</td>
<td>7</td>
<td>1</td>
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<td>4</td>
<td>5</td>
<td>6</td>
<td>5</td>
<td>0.33</td>
</tr>
</tbody>
</table>
## Monte Carlo Simulation Example

### Summary of Simulation Runs for Example Project

<table>
<thead>
<tr>
<th>Run Number</th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
<th>E</th>
<th>F</th>
<th>G</th>
<th>Critical Path</th>
<th>Completion Time</th>
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<tbody>
<tr>
<td>1</td>
<td>6.3</td>
<td>2.2</td>
<td>8.8</td>
<td>6.6</td>
<td>7.6</td>
<td>5.7</td>
<td>4.6</td>
<td>A-C-F-G</td>
<td>25.4</td>
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<tr>
<td>2</td>
<td>2.1</td>
<td>1.8</td>
<td>7.4</td>
<td>8.0</td>
<td>6.6</td>
<td>2.7</td>
<td>4.6</td>
<td>A-D-F-G</td>
<td>17.4</td>
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<tr>
<td>3</td>
<td>7.8</td>
<td>4.9</td>
<td>8.8</td>
<td>7.0</td>
<td>6.7</td>
<td>5.0</td>
<td>4.9</td>
<td>A-C-F-G</td>
<td>26.5</td>
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<tr>
<td>4</td>
<td>5.3</td>
<td>2.3</td>
<td>8.9</td>
<td>9.5</td>
<td>6.2</td>
<td>4.8</td>
<td>5.4</td>
<td>A-D-F-G</td>
<td>25.0</td>
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<td>4.5</td>
<td>2.6</td>
<td>7.6</td>
<td>7.2</td>
<td>7.2</td>
<td>5.3</td>
<td>5.6</td>
<td>A-C-F-G</td>
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<td>5.8</td>
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<td>2.8</td>
<td>5.2</td>
<td>A-C-F-G</td>
<td>22.3</td>
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<td>4.7</td>
<td>8.9</td>
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<td>7.3</td>
<td>4.6</td>
<td>5.5</td>
<td>A-C-F-G</td>
<td>24.2</td>
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<td>8.9</td>
<td>4.0</td>
<td>6.7</td>
<td>3.0</td>
<td>4.0</td>
<td>A-C-F-G</td>
<td>22.1</td>
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<td>1.1</td>
<td>7.4</td>
<td>5.9</td>
<td>7.9</td>
<td>2.9</td>
<td>5.9</td>
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<td>7.1</td>
<td>3.1</td>
<td>4.3</td>
<td>A-C-F-G</td>
<td>19.7</td>
</tr>
</tbody>
</table>
Project Duration Distribution

![Bar chart showing the distribution of project durations. The x-axis represents project length, ranging from 17 to 29, and the y-axis represents frequency, ranging from 0 to 10. The chart highlights the frequency of projects at different lengths, with a peak around project length 23.]
Probability

\[ P(X \leq \tau) = \frac{\text{Number of Times Project Finished in Less Than or Equal to } t \text{ weeks}}{\text{Total Number of Replications}} \]

The Probability that the project ends in 20 weeks or less is

\[ P \left( X \leq 20 \right) = \frac{13}{50} = 26\% \]
Criticality Index

- **Definition:** Proportion of runs in which the activity was in the critical path
- **PERT, CPM** assume binary (either 100% or 0%)
- Helpful for prioritizing effort in
  - Monitoring
  - Controlling
How Many Runs are Needed?

Criticality Index $p$ (particular node)

- Originally very conservative (10K runs)
- Empirical tests suggest $\leq 1000$ runs adequate
- Estimate of confidence interval for criticality

  - $(1-\alpha)$ confidence interval = symmetric interval around $\hat{p}$ such that $P$(true value $p$ is within that interval) is $(1-\alpha)\%$

$$\left(\hat{p} - Z_{\frac{\alpha}{2}} \sqrt{\frac{\hat{p}(1-\hat{p})}{n}}, \hat{p} + Z_{\frac{\alpha}{2}} \sqrt{\frac{\hat{p}(1-\hat{p})}{n}}\right)$$

- Consider a 95% confidence interval with $10\% \leq p \leq 90\%$, $400 \leq n \leq 1000$. Then with 95% confidence, $\hat{p}$ will be within 2%-5% of $p$
How Many Runs are Needed?

Mean Project Duration

- Must make assumptions regarding coefficient of variation \( \sigma / \mu \) (i.e. Std Dev/Mean)

- Basic formula 
  \[ \pm \text{Error \% } \approx 100 \frac{\sigma Z_\alpha}{\mu \sqrt{n}} \]

- For Empirical range of CoV (5\%..15\%)
  - Sample size 400: within .5\% to 1.5\% of true value \( \mu \)
  - Sample size 1000: within .3\% to 1\% of true value \( \mu \)

- Note inverse-root relationship: Halving error requires increasing # of trials by a factor of 4!
How Many Runs are Needed?
Project Duration Standard Deviation

- Basic formula \( \pm \text{Error } \% \approx \frac{100Z_{\alpha}}{\sigma \sqrt{2n}} \)
- Sample size 400: \( \hat{\sigma} \) within 7% of true value \( \sigma \)
- Sample size 1000: \( \hat{\sigma} \) within 4.38% of true value \( \sigma \)
- Inverse-root relationship again present
Monte Carlo: Summary

- Conceptually simple
  - Standard CPM used
  - No need for special assumptions about functional form of distributions
- Provides criticality index (valuable prioritization)
- Scalable analysis quality (albeit with super-linear effort required to reduce error)
- Computationally expensive
- Estimation of duration distributions can be expensive
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(Dynamic) Simulation Approach

- CPM-Based methods use simple representations
  - One-pass: No iteration
  - Represented uncertainty only with respect to *duration*
- Explicitly representing *process* brings benefits
  - Reasoning about process design
  - Identifying *emergent behavior* (e.g. dynamic bottleneck)
  - Simpler estimation of some uncertainties
- Must be clear about whether representations are just *process*-level or also *project*-level
Detailed Representation

- Repetitive processes for which aggregate representation is not desirable
- Processes where *static* planning is not possible
  - Repetitive processes for which # cycles unknown
  - Scheduling and coordinating complex interactions
    (Large #s of brief interactions, dependent on timing)
  - Cases where timing uncertainties change schedule
- Cases where individual timing component can be estimated, but where aggregate stats not known
Examples of Repetitive Processes

- Earth moving
- Tunneling
- Hotel/Apartment/Dormitory construction
- Road/Bridge construction
- Plumbing and glazing in high-rise
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- **Monte Carlo**

- **Simulation approaches**
  - General
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Simulation Example: Excavation and Transporting

- **Given**
  - **Front-end loader**
    - Output: $o_{\text{front-end loader}}$
    - Instantaneous time between loads
  - **Trucks**
    - $n$ vehicles
    - Capacity $c$
    - Load time $t_l$
    - Instantaneous dump time
    - Fully loaded speed $s_l$, empty speed $s_e$
    - Distance to dumpsite $d$

- **Naïve productivity:** $\min(o_{\text{front-end loader}}, o_{\text{trucks}})$