1.050 Engineering Mechanics I

Lecture 32
Energy bounds in beam structures (cont’d) - How to solve problems

1.050 – Content overview

I. Dimensional analysis
1. On monsters, mice and mushrooms
2. Similarity relations: Important engineering tools

Lectures 1-3
Sept.

II. Stresses and strength
3. Stresses and equilibrium
4. Strength models (how to design structures, foundations... against mechanical failure)

Lectures 4-15
Sept./Oct.

III. Deformation and strain
5. How strain gages work?
6. How to measure deformation in a 3D structure/material?

Lectures 16-19
Oct.

IV. Elasticity
7. Elasticity model – link stresses and deformation
8. Variational methods in elasticity

Lectures 20-32
Oct./Nov.

V. How things fail – and how to avoid it
9. Elastic instabilities
10. Plasticity (permanent deformation)
11. Fracture mechanics

Lectures 33-37
Dec.

Review: 3D isotropic elasticity

\[-e_{\text{com}} (\sigma') \leq \begin{cases} 
\max \left(-e_{\text{com}} (\sigma') \right) \\
\min_{\xi^*} e_{\text{pot}} (\xi') 
\end{cases} \leq e_{\text{pot}} (\xi') \]

Lower bound

Complementary energy approach

\[e_{\text{com}} (\sigma') = \psi (\sigma') - W^* (\ddot{u})\]

Solution

Potential energy approach

\[e_{\text{pot}} (\xi') = \psi (\xi') - W (\ddot{u})\]
Beam structures (2D)

Complementary free energy

\[ \psi^* = \int_{x=0}^{L} \left[ \frac{1}{2} \frac{N^2}{ES} + \frac{1}{2} \frac{M_y^2}{EI} \right] dx \]

Free energy

\[ \psi = \int_{x=0}^{L} \left[ \frac{1}{2} ES \left( \varepsilon_{xx}^0 \right)^2 + \frac{1}{2} EI \left( \theta_y \right)^2 \right] dx \]

Note: For 2D, the only contributions are axial forces & moments and axial strains and curvatures (general 3D case see manuscript page 263 and following)

Clapeyron's formulas

\[ \psi = \psi^* - \frac{1}{2}(W^* + W) \]
\[ \varepsilon_{\text{pot}} = \frac{1}{2}(W^* - W) \]
\[ \varepsilon_{\text{com}} = \frac{1}{2}(W - W^*) \]

Significance: Calculate solution potential/complementary energy ("target") from BCs

Beam structures

External work by prescribed displacements

\[ W^* = \sum_{j} \left[ \tilde{\varepsilon}_j^d(x_j) \cdot \tilde{R}_j(x_j) + \tilde{\omega}_j^y(x_j) \tilde{M}_{y,j} \right] \]

External work by prescribed force densities/forces/moments

\[ W = \int_{x=0}^{L} \tilde{\varepsilon}_j^d(x) dx + \sum_{j} \left[ \tilde{\varepsilon}_j^d(x_j) \tilde{F}_j(x_j) + \tilde{\omega}_j^y(x_j) \tilde{M}_{y,j} \right] \]

Beam elasticity

\[ -\varepsilon_{\text{com}} (\tilde{F}_j^*, \tilde{M}_{y,j}^*) \leq \max_{N,M_y,S.A.} \left( -\varepsilon_{\text{com}} (\tilde{F}_j^*, \tilde{M}_{y,j}^*) \right) \]
\[ \leq \min_{K.A.} \varepsilon_{\text{pot}} (\tilde{\varepsilon}_j^d, \tilde{\omega}_j^y) \]

Lower bound

Complementary energy approach
"Stress approach"
\[ \text{Solution} \]
Work with unknown but S.A. moments and forces

Upper bound

Potential energy approach
"Displacement approach"
\[ \text{Solution} \]
Work with unknown but K.A. displacements
Step-by-step solution approach

- **Step 1:** Express target solution (Clapeyron’s formulas) – calculate complementary energy AT solution
- **Step 2:** Determine reaction forces and reaction moments
- **Step 3:** Determine force and moment distribution, as a function of reaction forces and reaction moments (need \( M_i \) and \( N \))
- **Step 4:** Express complementary energy as function of reaction forces and reaction moments (integrate)
- **Step 5:** Minimize complementary energy (take partial derivatives w.r.t. all unknown reaction forces and reaction moments and set to zero); result: set of unknown reaction forces and moments that minimize the complementary energy
- **Step 6:** Calculate complementary energy at the minimum (based on resulting forces and moments obtained in step 5)
- **Step 7:** Make comparison with target solution = find solution displacement

Example

**Step 1:** Target solution
\[ \varepsilon_{com} = \frac{1}{2} \psi \delta \]

**Step 2:** Determine hyperstatic forces and moments (here: \( R' \))

**Step 3:** Determine force and moment distribution (as a function of hyperstatic force \( R' \)):
\[
M_i(x) = \begin{cases} 
\frac{P(1 - 2x)}{2} - R'(1 - \frac{x}{2}) & \text{if } 0 \leq x \leq 1/2 \\
- R'(1 - \frac{x}{2}) & \text{if } 1/2 < x \leq 1
\end{cases}
\]

**Note:** Only need expression for \( N \) and \( M_i \)

**Step 4:** Express complementary energy
\[
\varepsilon_{com} = \psi^* - W^* = \int \left( \frac{1}{2} N \frac{d}{dx} + \frac{1}{2} M_i \frac{d^2}{dx^2} \right) dx + \frac{1}{5} P^2
\]

Structure is statically indeterminate to degree 1
Can not be solved by relying on static equilibrium only (too many unknown forces, ‘hyperstatic’).

Goal: Solve problem using complementary energy approach
Example

Step 5: Find min of $\varepsilon_{\text{com}}(R')$

\[
\frac{\partial \varepsilon_{\text{com}}(R')}{\partial R} = 0
\]

\[
\frac{1}{2EI} \left( \frac{2l'}{3} l' P - \frac{5}{24} l' P \right) = 0
\]

\[
R' = \frac{5}{16} P
\]

Step 6: Minimum complementary energy

\[
\varepsilon_{\text{com}}(R' = \frac{5}{16} P) = \frac{7}{1536EI} l'^2 P
\]

Example

Step 7: Compare with target solution

\[
\varepsilon = \frac{1}{2} P S \leq \varepsilon_{\text{com}}(R' = \frac{5}{16} P) = \frac{7}{1536EI} l'^2 P
\]

\[
\delta = \frac{7}{768EI} l'^2 P
\]

\[
\text{represents a minimum of the complementary energy}
\]

Is it a global minimum, that is, the solution?

1. $M'$ is S.A.
2. $R'$ is the only hyperstatic reaction force (in other words, the only source of additional moments)
3. Therefore, the minimum is actually a global minimum, and therefore, it is the solution

Generalization (important)

- For any homogeneous beam problem, the minimization of the complementary energy with respect to all hyperstatic forces and moments

\[
X_i = [R_i, M, k_i]
\]

yields the solution of the linear elastic beam problem:

\[
\frac{\partial}{\partial X_i} (\varepsilon_{\text{com}}(X_i)) = 0
\]

\[
\frac{1}{2} (W - W^*) = \min_{X_i} \varepsilon_{\text{com}}(X_i)
\]