# 1.050 Engineering Mechanics I

## Review session

## 1.050 – Content overview

<table>
<thead>
<tr>
<th>I. Dimensional analysis</th>
<th>Lectures 1-3</th>
<th>Sept.</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. On monsters, mice and mushrooms</td>
<td>Lectures 4-15</td>
<td>Sept./Oct.</td>
</tr>
<tr>
<td>II. Stresses and strength</td>
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<tr>
<td>4. Strength models (how to design structures, foundations... against mechanical failure)</td>
<td>Lectures 33-37</td>
<td>Dec.</td>
</tr>
<tr>
<td>III. Deformation and strain</td>
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<tr>
<td>5. How strain gages work?</td>
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<td>6. How to measure deformation in a 3D structure/material?</td>
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<td>IV. Elasticity</td>
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<td></td>
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<tr>
<td>7. Elasticity model – link stresses and deformation</td>
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<tr>
<td>8. Variational methods in elasticity</td>
<td></td>
<td></td>
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<td>V. How things fail – and how to avoid it</td>
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<tr>
<td>9. Elastic instabilities</td>
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<td>10. Fracture mechanics</td>
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<td>11. Plasticity (permanent deformation)</td>
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</tbody>
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Notes regarding final exam

• Please contact me or stop by at any time for any questions

• The final will be comprehensive and cover all material discussed in 1.050. Note that the last two p-sets will be important for the final.
  – To get an idea about the style of the final, work out old finals and the practice final
  – There will be 2-3 problems with several questions each (e.g. beam problem/truss problem, continuum problem)
  – We will post old final exams from 2005 and 2006 today
  – We will post an additional, new practice final exam on or around Wednesday next week
  – Another list of variables and concepts will be posted next week
  – Stay calm, read carefully, and practice time management

Stress, strain and elasticity - concepts
### Overview: 3D linear elasticity

**Statically admissible (S.A.)**

- BCs on boundary of domain $\Omega$
  - $\partial \Omega_{\text{bd}} : \vec{T}^d (\vec{n}) = \vec{\sigma} \cdot \vec{n}$
- $\Omega : \begin{cases} \vec{T}(\vec{n}) = \vec{\sigma} \cdot \vec{n} \\ \text{div} \vec{\sigma} + \rho \vec{g} = 0 \\ \sigma_{ij} = \sigma_{ji} \end{cases}$

**Elasticity**

- Basis: Thermodynamics
  - $\vec{\sigma} = C : \vec{\varepsilon}$
  - $\sigma_{ij} = c_{ijkl} \varepsilon_{kl}$
  - $\vec{\sigma} = \left( K - \frac{2}{3} G \right) \varepsilon_{,1} + 2G \varepsilon$

### Isotropic elasticity

- Linear isotropic elasticity
- Written out for individual stress tensor coefficients
- Linear isotropic elasticity
- Written out for individual stress tensor coefficients, collect terms that multiply strain tensor coefficients

<table>
<thead>
<tr>
<th>$\sigma_{11}$</th>
<th>$\sigma_{12}$</th>
<th>$\sigma_{13}$</th>
<th>$\sigma_{22}$</th>
<th>$\sigma_{23}$</th>
<th>$\sigma_{33}$</th>
</tr>
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<tbody>
<tr>
<td>$\left( K - \frac{2}{3} G \right) \varepsilon_{11}$ + $2G \varepsilon_{11}$</td>
<td>$\left( K - \frac{2}{3} G \right) \varepsilon_{12}$ + $2G \varepsilon_{12}$</td>
<td>$\left( K - \frac{2}{3} G \right) \varepsilon_{13}$ + $2G \varepsilon_{13}$</td>
<td>$\left( K - \frac{2}{3} G \right) \varepsilon_{22}$ + $2G \varepsilon_{22}$</td>
<td>$\left( K - \frac{2}{3} G \right) \varepsilon_{23}$ + $2G \varepsilon_{23}$</td>
<td>$\left( K - \frac{2}{3} G \right) \varepsilon_{33}$ + $2G \varepsilon_{33}$</td>
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- General isotropic solid

**Strain tensor $\vec{\varepsilon}(\vec{x})$**

- Basis: Geometry
  - BCs on boundary of domain $\Omega$
    - $\partial \Omega_{\text{bd}} : \vec{\varepsilon}^d = \vec{\varepsilon}$
  - Linear deformation theory
    - $\vec{\varepsilon} = \frac{1}{2} \left( \text{grad} \vec{\varepsilon} + \left( \text{grad} \vec{\varepsilon} \right)^T \right)$
    - $\varepsilon_{ij} = \frac{1}{2} \left( \frac{\partial \varepsilon_{i}}{\partial x_{j}} + \frac{\partial \varepsilon_{j}}{\partial x_{i}} \right)$
<table>
<thead>
<tr>
<th>Variable</th>
<th>Definition</th>
<th>Notes &amp; comments</th>
</tr>
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<tbody>
<tr>
<td>$\nu$</td>
<td>$\varepsilon_{yy} = \varepsilon_{yy} = -\nu \varepsilon_{ss}$ $\nu = \frac{1}{2} \frac{3K - 2G}{3K + G}$</td>
<td>Poisson’s ratio (lateral contraction under uniaxial tension)</td>
</tr>
<tr>
<td>$E$</td>
<td>$\frac{9K}{3K + G}$ $\sigma_{ss} = E \varepsilon_{ss}$</td>
<td>Young’s modulus (relates stresses and strains under uniaxial tension)</td>
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**Uniaxial beam deformation**

F → $\sigma = \frac{F}{A}$ $x \rightarrow \varepsilon = \frac{x}{L}$

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**Solving problems with strength approach**

Use conditions for S.A. plus strength criterion (S.C.)
### Variable Definitions and Notes

<table>
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| Two pillars of stress-strength approach | At any point, \( \sigma \) must be:  
1. Statically admissible (S.A.)  
2. Strength compatible (S.C.) | - Equilibrium conditions “only” specify statically admissible stress field, without worrying about if the stresses can actually be sustained by the material – S.A.  
From EQ condition for a REV we can integrate up (upscale) to the structural scale  
**Examples:** Many integrations in homework and in class; Hoover dam etc.  
- Strength compatibility adds the condition that in addition to S.A., the stress field must be compatible with the strength capacity of the material – S.C.  
In other words, at no point in the domain can the stress vector exceed the strength capacity of the material  
**Examples:** Sand pile, foundation etc. – Mohr circle |

<table>
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<tr>
<td>( D_x )</td>
<td>( \forall {F_x, M_x} \in D_x(x) \Rightarrow f(x, F_x, M_x) \leq 0 )</td>
<td>Strength domain for beams</td>
</tr>
</tbody>
</table>
| \( |M_y|_{\text{lim}} = M_b \) | \( |M_y|_{\text{lim}} = M_b = \frac{1}{4} \sigma_b b h^2 \)  
For rectangular cross-section \( b, h \) | Moment capacity for beams |
| \( |N|_{\text{lim}} = N_b \) | \( |N|_{\text{lim}} = N_b = b h \sigma_b \) | Strength capacity for beams |
| \( f(M_x, N_x) \leq 0 \) | \( f(M_x, N_x) = \frac{|M_x|}{M_b} + \left( \frac{|N_x|}{N_b} \right)^2 - 1 \leq 0 \)  
\( \frac{|M|}{\sigma_b b h} \) \( 1 \) \( \frac{|N_x|}{\sigma_b b h} \) | M-N interaction (linear) \nM-N interaction (actual); convexity |
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<tr>
<td>Safe strength domain</td>
<td>$\forall i; \ 0 \leq Q^{(i)} \leq \alpha_i Q^{(i)}_{\text{lim}}$</td>
<td>Linear combination is safe (convexity)</td>
</tr>
<tr>
<td></td>
<td>$\sum_{i=1}^{\text{n}} \frac{Q^{(i)}}{Q^{(i)}<em>{\text{lim}}} \leq \sum</em>{i=1}^{\text{n}} \alpha_i = 1$</td>
<td></td>
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<tr>
<td></td>
<td>$Q^{(i)}_{\text{lim}}$: load bearing capacity of $i$-th load case</td>
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**Mohr circle**

Display 3D strain tensor in 2D projection – enables us to ‘see’ largest shear stresses, largest normal stresses…

Thereby facilitates application of strength criterion
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<tr>
<td>$D_k$</td>
<td>$\forall x : \sigma(x) \in D_k \iff f(x, \sigma(x)) \leq 0$</td>
<td>Strength domain (general definition) Equivalent to condition for S.C.</td>
</tr>
<tr>
<td>$D_{k,Tresca}$</td>
<td>$\forall x : f(x) =</td>
<td>v</td>
</tr>
<tr>
<td>$D_{k,Tension-cutoff}$</td>
<td>$\forall x : f(x) = \sigma - c \leq 0$</td>
<td>Tension cutoff criterion</td>
</tr>
<tr>
<td>$D_{k,Mohr-Coulomb}$</td>
<td>$\forall x : f(x) =</td>
<td>v</td>
</tr>
<tr>
<td>$\alpha_{lim}$</td>
<td>$\theta$</td>
<td>Angle of repose</td>
</tr>
<tr>
<td>Variable</td>
<td>Definition</td>
<td>Notes &amp; comments</td>
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</tr>
<tr>
<td>$S$</td>
<td>$S = \int_S dS$</td>
<td>Cross-sectional area</td>
</tr>
<tr>
<td>$I$</td>
<td>$I = \int_S z^2 dS$</td>
<td>Second order area moment</td>
</tr>
<tr>
<td>$EI$</td>
<td>$M = -EI \frac{d^2 \xi}{dx^2} = EI\vartheta$</td>
<td>Beam bending stiffness (relates bending moment and curvature)</td>
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</tbody>
</table>

\[
\frac{d^2 \xi}{dx^2} = \frac{f_x}{ES} \\
\frac{d^4 \xi}{dx^4} = \frac{f_z}{EI}
\]

- **Step 1**: Write down BCs (stress BCs and displacement BCs), analyze the problem to be solved (read carefully!)
- **Step 2**: Write governing equations for $\xi_1, \xi_2, \ldots$
- **Step 3**: Solve governing equations (e.g. by integration), results in expression with unknown integration constants
- **Step 4**: Apply BCs (determine integration constants)

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**Energy approach**

Approximate solution or find exact solution
Upper/lower bounds

\[-\mathcal{E}_{\text{com}}(\sigma') \leq \begin{cases} \max_{\sigma^{S.A.}} (-\mathcal{E}_{\text{com}}(\sigma')) \\ \text{is equal to} \\ \min_{\xi^{K.A.}} \mathcal{E}_{\text{pot}}(\xi') \end{cases} \leq \mathcal{E}_{\text{pot}}(\xi') \]

Lower bound
Complementary energy approach

Solution

Upper bound
Potential energy approach

\[
\mathcal{E}_{\text{com}}(\sigma') = \psi^*(\sigma') - W^*(\bar{T}') \\
\mathcal{E}_{\text{pot}}(\xi') = \psi(\xi') - W(\bar{\xi}')
\]

Boundary conditions

Important concept in 1.050 and elsewhere
Important BCs in beams/frames

Free end
\[ \tilde{F} = 0 \]
\[ \tilde{M} = 0 \]
\[ \xi_z = 0 \]
\[ \tilde{M}_y = 0 \]

Concentrated force
\[ Q_z = -P \]
\[ \xi_z = 0 \]
\[ \omega_y = 0 \]

Hinge (bending)
\[ M_y = 0 \]
\[ \xi = 0 \]
\[ \omega_y = 0 \]

Buckling of beams in compression

Elastic instability
Buckling

\[ P < P_{\text{crit}} = \frac{\pi^2EI}{(el)^2} \]

*el* 'effective length'

- Clamped cantilever beam
  \[ e = 2 \]
- Single supported beam
  \[ e = 1 \]
- Double clamped cantilever beam
  \[ e = \frac{1}{2} \]

Euler beam buckling
Different boundary conditions

Fracture mechanics

How to treat cracks in a continuum
Example: 3D fracture model

Expressions for $G$ can be found for a variety of geometries and structures:

For this geometry:

$$G = 1.12 \frac{2\pi a \sigma_0^2}{E} = 2\gamma_s$$

$$\sigma_0 = \frac{2\gamma_s E}{1.12 \pi a}$$

$$\sigma_0 = \frac{1}{1.12 \pi a} K_f$$

“Surface crack”

$\tilde{T}^d = \sigma_0 \bar{e}_z$ far away from crack

$\tilde{T}^d = -\sigma_0 \bar{e}_z$

Example application

Detect crack of length $a_m$

Concrete beam

Measure length of crack $a_m$

E.g. add fluorescent fluid

use UV light

http://www.amesresearch.com/images/cshst/block_before.jpg
Example application

Detect crack of length $a_m$

Concrete beam

Question: Will structure fail?

Solution: Solving beam problem provides us with stress distribution in section

$$\sigma_{xx}(z; x) = \frac{M_y(x)}{I} - z$$

$$\sigma_{xx}(z) = \frac{\sigma_0}{h/2} z$$

Calculate critical fracture stress and compare with stress in beam structure

$$\sigma_{0,\text{crit}} = \sqrt{\frac{1}{1.12^2 \pi a_m K_I}}$$

- If $\sigma_{xx}(z = -\frac{h}{2}) \geq \sigma_{0,\text{crit}}$, failure
- If $\sigma_{xx}(z = -\frac{h}{2}) < \sigma_{0,\text{crit}}$, no failure