NOTE

1. There are six problems of equal weight. Be sure to allocate an appropriate amount of time for each.
2. Solutions should be expressed in terms of the problem notation and then the numerical results should be obtained.
3. Please indicate clearly, using sketches when necessary, the assumptions and definitions you are introducing in carrying out your analyses. Do not hesitate to make reasonable assumptions, but state the reason why you make them.
4. Please be as neat as possible and clearly indicate what and where your answer is (only one answer!).
5. Unless noted otherwise, the fluid is water: \( (\rho = 1,000 \text{ kg/m}^3, \nu = 10^{-6} \text{ m/s}) \)
6. Cheat sheets #1, 2, and 3 are provided along with the Moody Diagram below.

Graph by MIT OCW.
A gate is built across a trapezoidal channel of cross-section shown in sketch. The gate holds back water that rises to an elevation of $h = 3.5\text{m}$ above the 8-m-wide horizontal bottom of the channel.

a) Determine the total pressure force, $P_{th}$, acting on the upstream, vertical face of the gate.

b) Determine the location of the center of gravity, in particular its elevation, $y_{CG}$, above the 8-m-wide horizontal bottom, of the trapezoidal face of the gate corresponding to $h = 3.5\text{m}$.

c) Determine the center of pressure, i.e. the line of action, for the total pressure force obtained in (a). [Note: Center of Pressure is not the same as Center of Gravity, since pressure varies linearly with depth, e.g. CG of a vertical rectangular area of height $h$ is $h/2$ above bottom, whereas the CP is $h/3$ above bottom.]
The sketch shows a section of pipe in which a nozzle-sleeve is inserted in the pipe. The pipe has a square cross-section with a sidelong of \( a_p = 10 \) cm, and the nozzle-sleeve reduces this sidelong to \( a_N = a_p/2 \). A mercury manometer \((\rho_m = 13.6\rho)\) is connected to the pipe as shown and gives a reading of \( h_m = 6.1 \) cm.

a) Determine the discharge, \( Q \), in the pipe. [Default value \( Q = 0.011 \text{ m}^3/\text{s} \)]

b) Determine the headloss associated with the nozzle-sleeve inserted in the pipe when wall-friction is neglected.

c) Determine the force exerted by the flow on the nozzle-sleeve (when wall-friction is neglected).

d) Assuming the pipe material to have a roughness \( \varepsilon = 0.05 \) mm, determine the length of pipe required to give a frictional headloss equal to the headloss computed in (b), and comment on why this length justifies the neglect of wall friction in (b) and (c).
The sketch (not to scale) shows a pond that is connected to a lake through a horizontal $\ell = 400$ m long straight circular concrete pipe, $\epsilon = 1$ mm. The pipe inlet and outlet are located some distance above bottom, to prevent erosion. The pond receives an inflow of water, $Q_p = 0.25$ m$^3$/s, and it is desired to maintain the pond at a level, $h_p$ of 1.0 m above the lake level, $h_L$, by proper choice of the diameter $D$ of the concrete pipe.

a) Set up a relationship (in general terms, i.e. using $D$ for diameter, $f$ for friction factor, $K_L$ for minor loss coefficients, etc.) between the velocity $V$ in the pipe and the difference in water levels, $h_p - h_L$.

b) Determine the pipe diameter $D$ which would give the required flow, $Q$, from pond to lake for $h_p - h_L = 1$ m.

c) Corresponding to your solution in (b) sketch the Energy and Hydraulic Grade Lines from inlet to exit of the pipe. Although you are asked for a sketch, try to make it to scale in the vertical.

d) A graduate of Haavad College suggests that a smaller diameter would be required if the pipe was installed with a slope from the pond towards the lake instead of being horizontal. Is s/he correct?
A pump (at elevation $z_p = 0$) delivers a flow rate of $Q = 0.1 \text{ m}^3/\text{s}$ from a reservoir ($z_r = 5 \text{ m}$) to a water tank (the "user") through a 25-cm-diameter cast iron pipe ($d = 0.4 \text{ mm}$). The water level in the tank is at $z_u = 100 \text{ m}$ with the pipe outlet 5 m above the free surface. The total length of pipe connecting reservoir and user is 300 m and minor losses may be neglected. The pipe passes over the top of a hill ($z_t = 110 \text{ m}$) at a distance of 70 m from the user.

a) Determine the velocity in the pipe.

b) Determine the Darcy-Weisbach friction factor, $f$.

c) Determine the power (1 HP = 745 watt) required by the pump motor if the pump's efficiency is assumed to be $\eta = 0.8$.

d) Determine the pressure in the pipe at its highest elevation. $z_t = 110 \text{ m}$.

e) Is the pressure determined in (d) of concern, i.e. is cavitation a potential problem, and what would happen if the pipe developed a leak at its highest point?
Typical rivers are much wider than they are deep. The sketch above shows a typical river crosssection, for which the slope of the river banks, i.e. the lateral slope, is denoted by $a$ and as a consequence of $h<<b$ we have that $a$ is small, say $a < 10^\circ$.

a) By balancing gravitational and frictional forces for a uniform steady flow in a prismatic channel of slope $S_o$, derive the classical hydraulic formula for the boundary shear stress $\tau = \rho g R_h S_o$.

b) Show that the hydraulic radius, $R_h$, and the mean depth, $h_m$, are virtually identical for the typical river crosssection discussed above.

c) Assuming $R_h = h_m$ and adopting the Darcy-Weissbach expression for the uniform steady ("normal") flow velocity in a channel of slope $S_o$, obtain a simple expression for the Froude Number in terms of Darcy-Weissbach's $f$ and the slope $S_o$ corresponding to normal flow.

d) From the result obtained in (c), obtain a rough estimate of the bottom slope, $S_{oc}$, for which normal flow in a "typical river" will be critical.
Problem No. 6

The sketch shows the cross section of a trapezoidal channel with a bottom width, $b = 8$ m, and sides sloping 1 on 1. The channel roughness is estimated to be of the order $e = 2$ cm, and its normal depth, corresponding to uniform steady flow, is $h_n = 3.5$ m. The channel is carrying a discharge of $Q = 200$ m$^3$/s.

a) What is the value of Manning's "n" for this channel? (Default value is "n" = 0.02 in SI-units)

b) Estimate the slope of the channel, $S_0$.

c) Is normal flow super or subcritical? (Justify your answer)

d) Corresponding to normal flow determine the Specific Energy [Head], $E_o$.

e) Corresponding to normal flow determine the Momentum and Pressure Thrust, $MP$. [take advantage of the similarity between this problem and Problem No. 1]
Problem No. 1

(a) Splitting the cross-section into a rectangular area ($A_2$) and two identical triangular areas ($A_1$ and $A_3$) we have:

- $A_1 = A_3 = \frac{1}{2} h^2$
- $C_{G_1} = C_{G_2} = \frac{1}{3} h$ below free surface
- $A_2 = b \cdot h$
- $C_{G_2} = \frac{1}{2} h$ below free surface

\[
P_h = \sum P_c \cdot A = 2 \cdot \left(\rho g \frac{1}{3} h\right)\frac{1}{2} h^2 + \left(\rho g \frac{1}{2} h\right) b h - \rho g \left[\frac{1}{3} h^3 + \frac{1}{2} h^2 b\right] = 1000 \cdot 9.8 \left[\frac{1}{3} \cdot 3.5^3 + \frac{1}{2} \cdot 3.5^2 \cdot 8\right] = 620 \text{ kN}
\]

(b) The general formula for total hydrostatic pressure for heads: $P_h = \rho g (h - y_{G}) A$, or

\[
h - y_{G} = \frac{P_h}{\rho g A} = \frac{620 \text{ kN}}{1000 \cdot 9.8 \left[2 \frac{1}{2} \cdot 3.5^2 + 3.5 \cdot 8\right]} = 1.57 \text{ m} \Rightarrow y_{G} = 1.93 \text{ m}
\]

$y_{G}$ is measured from the 8-m-wide bottom upwards.
Note: Same result is obtained by applying the definition of Center of Gravity of an area:
\[ Y_{CG} = \frac{\Sigma (y_i A_i)}{\Sigma A} = \frac{[2 \cdot \frac{2}{3} h \cdot \frac{1}{2} h^2 + \frac{1}{2} h \cdot h b]}{[2 \cdot \frac{1}{2} h^2 + h b]} = 1.93 \text{ m} \]

For area \( A_2 \), which is rectangular, we have the standard 2-D formula for the moment:
\[ M_2 = P \cdot \frac{1}{2} h = \frac{1}{2} \rho g h^2 b \cdot \frac{1}{2} h = 560 \text{ kN m} \]
[Note: \( CG_2 = \frac{1}{2} h \) above bottom, but \( CP_2 = \frac{1}{3} h \) above bottom, i.e. Center of Gravity \( \neq \) Center of pressure]

For area \( A_1 \) (same as \( A_3 \)), which is triangular, standard 2-D formula does not apply! We must use fundamentals.

\[ p(y) = \rho g (h-y) \quad ; \quad b(y) = y \quad ; \quad \text{arm} = y \]

\[ y \]
\[ M_1 = \int (p(y) \cdot b(y) \cdot y) dy = \rho g \int (h y^2 - y^3) dy = \rho g \left[ \frac{1}{3} h^3 y^3 - \frac{1}{4} y^4 \right]_0^h - \rho g \frac{1}{12} h^4 = 122.6 \text{ kN m} \]

So, we have by definition of Center of Pressure:
\[ Y_{Cp} = \frac{\Sigma M}{P_{\text{hy}}} = \frac{560 + 2 \cdot 122.6}{620} - \frac{805}{620} = 1.3 \text{ m} \]

(above bottom, located on \& of crosssection, and acting horizontally towards the dam).

[Note: For triangular areas, the Center of Pressure is: \( Y_{CP1} = Y_{CP2} = M_1 / (\rho g \cdot \frac{1}{3} h \cdot \frac{1}{2} h^2) = 122.6 / 70 = 1.75 \text{ m} \), i.e. \( CP \) is at mid-depth, \( \frac{1}{3} h \) above bottom, whereas \( CG \) is \( (\frac{2}{3})h \) above bottom : \( Y_{CP} + Y_{CG} \)]
Problem No. 2

Between upstream and nozzle opening, we have a short transition of a converging flow. No headloss. Therefore, with \( A_p = \text{pipe area} = Q_p^2 \) [It's a square pipe!] and \( A_n = Q_n^2 = (Q_p/2)^2 = \frac{1}{4} Q_p^2 = \frac{1}{4} A_p \), application of conservation of mass gives

\[
Q = V_p A_p = V_n A_n \Rightarrow V_n = 4 Q_p = 4 \frac{Q_p}{A_p}
\]

and Bernoulli

\[
\frac{V_p^2}{2g} + P_p/\rho g + Z_p = \frac{V_n^2}{2g} + P_n/\rho g + Z_n + \frac{1}{2} \frac{A}{L}
\]

or

\[
(P_p + \rho g Z_p) - (P_n + \rho g Z_n) = \frac{1}{2} \rho (V_n^2 - V_p^2) = \frac{15}{2} \rho V_p^2
\]

Manometers of the type shown record difference in piezometric head, or

\[
(P_p + \rho g Z_p) - (P_n + \rho g Z_n) = (\rho_m - \rho) g h_m
\]

Introducing this in expression from Bernoulli gives
\[(p_m - P)g \cdot h_m = \frac{15}{2} P \cdot V_p^2 = V_p = \sqrt{\frac{2}{5}} (p_m/p - 1)g \cdot h_m\]

and with \(p_m/p = 13.6\), and \(h_m = 5.3\text{cm} = 0.0053\text{m}\), we get

\[V_p = 1.002 = 1.00\text{m/s}; \quad Q = A_p \cdot V_p = q_p \cdot V_p = 0.01\text{m}^3/\text{s}\]

b) Following outflow from the nozzle opening the flow expands. This results in an expansion head loss, given by

\[\Delta H_{\text{exp}} = \frac{(V_N - V_p)^2}{2g} = \frac{(4V_p - V_p)^2}{2g} = \frac{3^2}{2g} = 0.46\text{m}\]

c) Neglecting friction, Bernoulli from upstream of nozzle-sleeve to far enough downstream to make flow well-behaved gives

\[\frac{V_p^2}{2g} + \frac{p_p}{pg} + \frac{\gamma}{g} = \frac{V_p^2}{2g} + \frac{p_d}{pg} + \frac{\gamma}{g} + \Delta H_{\text{exp}} + \Delta H_f\]

\[p_p - p_d = pg \cdot \Delta H_{\text{exp}}\]

Applying the momentum principle to same control volume, we have

\[(\vec{S} \cdot V_p + p_p) \cdot A_p = \vec{M}_p = (\vec{S} \cdot V_p^2 + p_d) \cdot A_p + \vec{F}_s\]

\[\vec{F}_s = (p_p - p_d) \cdot A_p = pg \cdot A_p \cdot \Delta H_{\text{exp}} = 45.1\text{N}\]

This is the force on the nozzle-sleeve which is in direction of flow.
\( A_p = a_p^2 \); \( P = \text{wetted perimeter} = 4a_p \)
\( R_h = A_p / P = a_p / 4 \Rightarrow 4R_h = a_p \) [used for D]

\[ \text{Re} = \frac{4R_h V_p}{\nu} = \frac{a_p V_p}{\nu} = 10^5 \quad \eta/\eta_p = \frac{0.05}{100} = 0.0005 \]

Now, from Moody:
\[ f = 0.0205 \quad \text{(good eyes)} \]

Thus,
\[ \Delta H_f = f \frac{4L}{a_p} \frac{V_p^2}{g} = \Delta H_{\text{exp}} = 0.46 \text{ m} \]
\[ L_f = \Delta H_{\text{exp}} \frac{z g}{V_p^2} = 43.98 \approx 44 \text{ m} \]

44 m of pipe is required to give the same head loss as the nozzle causes. This 'enormous' length, compared to the length of the nozzle-affected flow (maybe of the order 10a_p = 1 m) justifies the negligible head loss assumed in (a) and also the neglect of shear stresses (frictional forces) in (c).
Problem NO: 3

\[ h_p - \frac{V_p^2}{2g} \]

\[ \frac{V_p^2}{2g} \]

\[ \Delta H_{\text{air}} = \frac{V_p^2}{2g} \]

\[ L = 400 \text{ m} \]

\[ \frac{V_p}{2g} \]

---

a)

Head losses between pond and lake consist of:
Frictional loss = \( f \left( \frac{L}{D} \right) \frac{V_p^2}{2g} \)
Enhance loss [re-entrant inflow, \( C_c = 0.5 \)] = \( K_{c,\text{ent}} \frac{V_p^2}{2g} \)
Exit loss to lake = \( K_{c,\text{exit}} \frac{V_p^2}{2g} \)

\[ K_{c,\text{ent}} = \left( \frac{1}{C_c} - 1 \right) = 1 \]

\[ K_{c,\text{exit}} = \left( 1 - \frac{g}{g_{\text{pipe}}} \right)^2 = 1 \] (Lake \( \approx \) Pipe)

Therefore:

\[ h_p = h_L + \left( K_{c,\text{ent}} + K_{c,\text{exit}} + f \frac{L}{D} \right) \frac{V_p^2}{2g} = h_L + \left( 2 + 400 \frac{f}{D} \right) \frac{V_p^2}{2g} \]

\[ V_p = \sqrt{2g(h_p - h_L)} \sqrt{2 + 400f/D} \quad \text{(Din meters)} \]

b)

\[ Q = \frac{V_p A}{\pi} \Rightarrow V_p = \frac{Q}{A} = \frac{4}{\pi} \frac{Q}{D^2} \Rightarrow \text{Info (a) gives} \]

\[ D^2 = \frac{4Q}{\pi} \sqrt{2 + 400f/D} \sqrt{2g(h_p - h_L)} \]

or, with \( Q = 0.25 \text{ m}^3/\text{s} \) and \( h_p - h_L = 1 \text{ m} \)

\[ D = 0.268 \left( 2 + 400f/D \right)^{1/4} \text{ (m/s)} \]

\[ D = 0.268 \left( 2 + 400f/D \text{ (m)} \right)^{1/4} \]
We must solve this by iteration, but neither \( f \) nor \( D \) is known. So, we start by taking \( f = 0.02 \) (our good old standby) and iterate to get \( D \) starting with \( D^0 = \infty \) (equivalent to neglect of friction)

\[
D^0 = \infty \Rightarrow D^{(1)} = 0.32 \text{ m} \Rightarrow D^{(2)} = 0.61 \text{ m} \Rightarrow D^{(3)} = 0.53 \text{ m} \Rightarrow D^{(4)} = 0.55 \text{ m}
\]

So, \( D = 0.54 \text{ m} \) if \( f = 0.02 \), but is it?

\[
Re = D \cdot \frac{V}{U} = D \left[ \frac{Q}{(g \cdot D^2)} \right] \cdot \frac{V}{U} = 0.84 \cdot 1.09 \cdot 10^{-6} = 5.9 \cdot 10^5
\]

\[
\frac{V}{D} = 0.001 \Rightarrow 0.54 = 1.9 \cdot 10^{-3}
\]

Moody Diagram gives: \( f = 0.0225 \). With this value, and starting iterations with \( D^{(4)} = 0.54 \text{ m} \), we obtain:

\[
D^{(5)} = 0.54 \Rightarrow D^{(6)} = 0.56 \text{ m} \Rightarrow D^{(7)} = 0.55 \text{ m} \Rightarrow D = 0.55 \text{ m}
\]

We are done: Change in \( D \) from 0.54 to 0.55 m will not produce a change in \( Re \) & \( \frac{V}{D} \) to change \( f \), so: \( f = 0.0225 \) and \( D = 0.55 \text{ m} \)

c)

For \( D = 0.55 \text{ m} \) we have \( V_p = \frac{Q}{(g \cdot D^2)} = 1.04 \text{ m} \) and therefore \( \frac{V_p^2}{2g} = 0.22 \text{ m} \). At vena contracta of inflow \( V_c = \frac{V_p}{C_c} = 2V_p \) and \( \frac{V_c^2}{2g} = 4 \cdot \frac{V_p^2}{2g} = 0.88 \text{ m} \).

\[
E_{GL} = H (m) \text{ above pipe } 2 \left( \frac{V_p}{2} - 0 \right) \}
\]

\[
E_{GL} = HGL = \left( H - \frac{V_p^2}{2g} \right) \text{ above pipe } 2
\]

For graphical representation see sketch at start of solution to this problem.

d)

So long as pipe exit is below free surface in lake, the exit head is \( h_e = \text{lake level} \Rightarrow \text{actual exit does not enter problem at all} \Rightarrow (\text{regardless of gender!}) \)

The Haaward student is wrong.
Problem NO: 4

Not to scale.
Lengths in meters

\[
\begin{align*}
Z_e &= 110 \text{ m} \\
5 \text{ m} &
\]

Neglect all minor losses

\[\times \text{20 m} \times \text{10 m}\]

a) 

\[
V = \frac{Q}{A} = 0.1/(\frac{\pi}{4} \times 0.25^2) = 2.04 \text{ m/s}
\]

b) 

\[
Re = \frac{VD}{V} = 2.04 \times 0.25 / 10^{-6} = 5 \times 10^5; \quad \frac{\varepsilon}{D} = \frac{0.20 \times 10^{-3}}{0.25} = 0.0016
\]

Moody gives : \[f = 0.022\]

c) 

Head at start = \(Z_r = 5\) \text{ m}; \quad \text{Head at end} = (Z_u + 5) + \frac{V^2}{2g} = 105.2 \text{ m}

\[H_p = \text{pump head} = 105.2 - 5 + \text{head loss} = 100.2 + f(\varepsilon/b)\frac{V^2}{2g} = 100.2 + 0.022 \times (300/0.25) \times 2.04^2 / (2 \times 9.8) = 105.8 \text{ m}\]

Power supplied from pump = \(pgQH_p = 103.7 \text{ kW}\)

Power to pump = \(8HP = \frac{pgQH_p}{\eta_p} = 103.7/0.8 = 129.6 \text{ kW} = 174 \text{ HP}\)

d) 

\[H_{hp} = \frac{V^2}{2g} + Z_e + \frac{P_t}{\rho g} = (Z_u + 5) + \frac{V^2}{2g} + f\frac{\varepsilon u}{D} \frac{V^2}{2g} = 105 + (14022 \times 0.25/2.5)^2\]

\[P_t = \rho g \left(105 - 110 + 0.022 \frac{70}{0.25 \times 2.98}\right) = \rho g (-5 + 1.31) = -3.69\rho g\]

\[P_t = -36.2 \text{ kPa (Gauge Pressure)}\]

e) 

\[
\frac{P_t}{\rho g} = -3.69 \text{ m} \Rightarrow \text{Not close to } -10 \text{ m} \Rightarrow \text{Cavitation not a problem}
\]

If pipe has leak at \(Z_c\) air will be sucked in since \(P_t < 0\) and flow may be disrupted.
Problem No. 5

a) Shear force parallel to bottom = $T_s R_s$

$A_s =$ area on which $T_s$ act = $P \Delta x = F_{T_s} = T_s P \Delta x$

Gravity force parallel to bottom = mass $\cdot g_x$

mass = $\rho A \Delta x$; $g_x = g \sin \beta = g S_o$; $[S_o = \sin \beta]$ →

$F_{g_x} = \rho g A S_o \Delta x$

$F_{T_s} = T_s P \Delta x = F_{g_x} = \rho g A S_o \Delta x \Rightarrow T_s = \rho g (\frac{A}{g}) S_o = \rho g R S_o$

b) $R_h =$ hydraulic radius = $A/P$

$h_m =$ mean depth = $A/b_s$

line-element along free surface, $\delta b_s$, is related to line-element along bottom, $\delta P$, by $\delta b_s = \cos \alpha \delta P$

If $\alpha < 10^\circ$ then $\cos \alpha > 0.985$, so $\delta b_s \approx \delta P$

$b_s = \Sigma \delta b_s \approx \Sigma \delta P = P \Rightarrow R_h \approx h_m$

c) $\tau_s = \rho g R_h S_o$ [from (a)]; $\tau_s = (f/8) g V^2$ [from Cheat Sheet]

$\rho g R_h S_o = \frac{f}{8} g V^2 \Rightarrow \frac{V^2}{g R_h} = \frac{V^2}{g h_m} = \tau r^2 = \frac{8 S_o}{f}$

d) If $\tau r = 1$ for normal flow, then

$\tau r = 1 = \frac{8 S_o}{f} \Rightarrow S_o = \frac{f}{8}$

With typical value for $f = 0.02$ we have

If $S_o < \frac{f}{8} \approx 2.5 \times 10^{-3}$ slope is steep

If $S_o > \frac{f}{8}$ slope is mild
Problem No: 6

Channel Crosssection is identical to the one discussed in Problem No: 1.

a) Area $A = bh + 2\frac{1}{2}h^2 = (b+h)h = 11.5 \cdot 3.5 = 40.25 \text{ m}^2$

Wetted Perimeter $P = b + 2\sqrt{2h} = 8 + 2.83 \cdot 3.5 = 17.9 \text{ m}$

Hydraulic Radius $R_h = \frac{A}{P} = \frac{40.25}{17.9} = 2.25 \text{ m}$

Manning's $n'' = 0.038$ (0.02)$^{1/6} = 0.0198$ (SI-units)

$V_n = \frac{Q}{A} = \frac{200}{40.25} = 4.97 m^3 = \frac{1}{h} R_h^{2/3} S_0^{1/2} = \frac{2.25^{2/3}}{0.0198} S_0^{1/2}$

$S_0 = \left[4.97 \cdot 0.0198 \left(2.25\right)^{2/3}\right]^2 = 3.28 \cdot 10^{-3}$

b) Mean depth $h_{mn} = \frac{A}{b_s} = \frac{A}{b+2h} = \frac{40.25}{8+2 \cdot 3.5} = 2.68 \text{ m}$

$F_r = \frac{V_n}{\sqrt{g h_{mn}}} = \frac{4.97}{9.8 \cdot 2.68} = 0.97 \leq 1$ (But not by a lot)

[Note: If $h_{mn}$ is incorrectly taken as $h=3.5 \text{ m}$, $F_r=0.85$]

$F_r < 1$ means that normal flow is subcritical

c) $E_{sn} = h + \frac{V_n^2}{2g} = 3.5 + \frac{4.97^2}{2 \cdot 9.8} = 3.5 + 1.26 = 4.76 \text{ m}$

d) From Problem No: 1 we have: $P = P_f = 620 \text{ kN}$

The momentum force is: $M = g V_n^2 A = 1000 \cdot 4.97^2 \cdot 40.25 = 994 \text{ kN}$

$MP = M + P = 994 + 620 = 1614 \text{ kN}$