Continuum Hypothesis

\[ \frac{\partial u}{\partial x} = \lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x} = \lim_{\Delta x \to 0^+} \frac{\Delta u}{\Delta x} \]

where "0" is a scale much smaller than any in which we are interested.

Compressibility of Fluids

\[ E \frac{\Delta V}{V} = -\nabla p \]

\[ E \frac{\Delta \varphi}{\varphi} = \nabla p \]

\[ E_v = \text{bulk modulus} \]

For water:

\[ E_v = 2.15 \times 10^9 \text{ N/m}^2; \quad \nabla p = 10^8 \text{ N/m}^2 (~10 \text{ km depth}) \]

\[ \left| \frac{\Delta V}{V} \right| \approx \left| \frac{\Delta \varphi}{\varphi} \right| \approx 5\% \quad \text{NOT MUCH} \]

For air:

\[ E_v = 1.4 \times 10^5 \text{ N/m}^2; \quad \nabla p = 10^4 \text{ N/m}^2 (~800 \text{ m height}) \]

\[ \left| \frac{\Delta V}{V} \right| \approx \left| \frac{\Delta \varphi}{\varphi} \right| \approx 7\% \quad \text{NOT MUCH} \]

\[ C = \text{speed of sound} = \sqrt{\frac{\nabla p}{\nabla \varphi}} = \sqrt{\frac{\nabla p}{\rho}} = \sqrt{\frac{E_v}{\rho}} \]

\[ \begin{cases} 1500 \text{ m/s (water)} \\ 335 \text{ m/s (air)} \end{cases} \]
If \( V = \text{fluid velocity} \ll c - \text{speed of sound in fluid} \), the fluid can be considered incompressible when pressure variations are not excessive.

**Fluid Velocity**

"Fluid particle" - a small volume \( \Delta V = 0 \) that consists of "the same" molecules.

\[
\mathbf{\dot{\mathbf{q}}} = \frac{\mathbf{q}(x_0, y_0, z_0, t)}{t, t_0}
\]

\[
\lim_{\Delta t \to 0} \frac{\Delta x, \Delta y, \Delta z}{\Delta t} = \left( u_0, v_0, w_0 \right)
\]

velocity (vector) at point \((x_0, y_0, z_0)\) at time \(t_0\)

**Choice of Coordinate System**

**Lagrangian Coordinates**

Identify a fluid particle by its position \( \mathbf{r}_0 \) at \( t = 0 \) and determine its position \( \mathbf{r}(t) \) at any subsequent time.

\[
\mathbf{r}(\mathbf{r}_0, t) = \mathbf{q} = \frac{d\mathbf{r}}{dt} = \frac{d\mathbf{q}}{dt} = \frac{d^2\mathbf{r}}{dt^2}
\]

(position, velocity, acceleration)
Eulerian Coordinates

Determine the velocity vector \( \vec{q} \), at a fixed point \((x, y, z)\), as a function of time:

\[
\vec{q}(x, y, z, t) = (u, v, w)
\]

This coordinate system - Eulerian - is favored over Lagrange's in Fluid Mechanics.

If \( \vec{q}(x, y, z, t) \) is not a function of time, i.e.

\[
\frac{\partial \vec{q}}{\partial t} = \left( \frac{\partial u}{\partial t}, \frac{\partial v}{\partial t}, \frac{\partial w}{\partial t} \right) = (0, 0, 0) = 0
\]

the flow is referred to as **STeady Flow**, but \( \frac{\partial \vec{q}}{\partial t} = 0 \) does not imply that the fluid is not accelerating.

\[
\vec{q}(x_0, y_0, z_0, t_0)
\]

\[
\vec{q}(x_1, y_1, z_1, t_1)
\]

Following the same fluid particle as we must according to Newton's Law, we have

\[
\dot{\vec{q}} = \frac{d\vec{q}}{dt} = \frac{D\vec{q}}{Dt} = \lim_{\Delta t \to 0} \frac{\vec{q}(x_1, y_1, z_1, t) - \vec{q}(x_0, y_0, z_0, t_0)}{t - t_0}
\]
or, with \( x_i - x_0 = u_0(t_i - t_0) \) and analogous we have

\[
\frac{d\vec{q}}{dt} = \frac{D\vec{q}}{Dt} = \lim_{\Delta t \to 0} \frac{\Delta q_u \Delta t + \Delta q_v \Delta t + \Delta q_w \Delta t + \Delta q_t \Delta t}{\Delta t} =
\]

\[
\frac{\partial}{\partial t} + (\vec{q} \cdot \nabla) \vec{q} = \frac{\partial q}{\partial t} + (\vec{q} \cdot \nabla) \vec{q}
\]

where

\( \nabla = \text{"del" operator} = (\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}) \)

In words:

Total derivative or Material Derivative = rate of change (in this case of velocity \( \vec{q} \)) following a fluid particle

Local rate of change, i.e. the rate of change taking place at the fixed location \((x, y, z)\). Note this would be zero if flow is steady

The convective rate of change, i.e. the rate of change associated with the particle moving to a new location where conditions (here the velocity) has changed relative to the original location. Note this term would be zero if \( \vec{q} \) independent of location: UNIFORM FLOW
Streamline

**Definition:** A streamline is a line that at a given instant of time has the local velocity vector as its tangent at any point along the line.

\[ \vec{q}(x, y, z, t_0) \]

\[ \vec{q}(x, y, z, t_0) \]

By definition, it therefore follows that

\[ ds = \text{infinitesimal element along streamline} = (dx_s, dy_s, dz_s) = \vec{q}(s, t_0) = (u_s, v_s, w_s) \]

or

\[ \frac{dx_s}{u_s} = \frac{dy_s}{v_s} = \frac{dz_s}{w_s} \]

If the flow is STEADY the velocity vector at any point does not vary with time, \( \frac{d\vec{q}}{dt} = 0 \). Hence the streamline is independent of time, and since a particle on a streamline always moves tangential to the line, it will follow a path (the **pathline**) equal to the streamline.