Conservation of Volume for a Streamtube
(Z-D Plane Flow)

Streamline coordinates:
\( \tilde{s} \) is local direction of streamline
\( \tilde{h} \) is local direction \perp \) streamline

and
\[
\tilde{q}(\tilde{s}) = (q_s, q_h) = (q_s, 0) = q_s
\]

STREAM FUNCTION

Define a function \( \Psi \) such that
\[
\frac{\partial \Psi}{\partial \tilde{s}} = 0 \quad , \quad \frac{\partial \Psi}{\partial \tilde{h}} = q_s
\]

and therefore
\[
\Psi = \Psi(\tilde{s}, \tilde{h}) = \Psi(h) = \text{constant along } \tilde{s}
\]

\( \Psi \) = stream function is constant along a streamline.
\[ \frac{\partial \psi}{\partial h} = \lim_{\Delta h \to 0} \frac{\Delta \psi}{\Delta h} = \frac{\psi_2 - \psi_1}{\Delta h} = q_s \]

\[ \psi_2 - \psi_1 = q_s \Delta h = Q_{12} \text{ - discharge in the 1-2} \]

**Velocity Potential**

Define a function \( \phi \) such that

\[ \frac{\partial \phi}{\partial s} = q_s \quad , \quad \frac{\partial \phi}{\partial h} = 0 \]

and therefore

\[ \phi = \phi(s, h) = \phi(s) = \text{constant along lines } L \]

\( \phi \) = velocity potential is constant along lines \( L \) streamlines (equipotential lines)
\[ \frac{d\Phi}{ds} = \lim_{\Delta S \to 0} \frac{\Delta \Phi}{\Delta S} = \frac{\Phi_2 - \Phi_1}{\Delta S} = \frac{Q_s}{\Delta S} \]

or

\[ \Phi_2 - \Phi_1 = Q_s \Delta S = (Q_s \Delta h) \frac{\Delta S}{\Delta h} = Q_{12} \]

if \( \Delta S/\Delta h = 1 \)

**Flow NET**

Lines of constant \( \Psi \) (streamlines) and constant \( \Phi \) (equipotential lines) form two families of curves (s-lines and h-lines) that intersect each other at right angles, i.e. they form a "rectangular" pattern.

If the two families of curves are constructed (drawn) such that they form a pattern of "squares", i.e. \( \Delta S = \Delta h \), the discharge in stream tubes formed by adjacent stream lines, \( Q_{12} \), is constant (as it should be). This is a **Flow NET**

If we know the discharge, \( Q_{12} \), in a stream tube, we can obtain an estimate of the velocity at the center of a "square" from \( Q_s = Q_{12} / \Delta h \), where \( \Delta h \) (\( \Delta S \)) is the side length of the square.
Simple "Rules" to follow when Constructing Flow Nets

1) Use a pencil and bring an eraser.

2) Solid boundaries are streamlines

3) Regions where flow is uniform - straight parallel streamline - is a good place to start and end.

4) Sketch the streamlines and equipotential lines connecting regions identified in (3) observing the following:
   - $\Psi$ and $\Phi$-lines form a "square" pattern
   - Streamlines can never cross
   - Streamlines can start and end only at flux areas
   - Streamlines in the interior are guided by boundaries, but smoother (no kinks)

5) Examine sketch and "repair" where needed (this is where the eraser comes in!)

6) Discharge $= \Delta Q$ is constant within a tube
   - $V = \Delta Q / \Delta h$ can be estimated at center of "squares". Directed along $\Psi$.

7) Pressure is estimated from Bernoulli along a streamline.
STREAM FUNCTION & VELOCITY POTENTIAL

Their Generalized Definitions and Limitations

Differential form of Continuity Equation

\[ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \] in 2-D

\[ \nabla \cdot \vec{v} = 0 \] in 3-D

Stream Function, \( \psi \)

\[ \psi = \psi (x, y, t) \]

defined by

\[ u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x} \]

Continuity equation (in 2-D) is automatically satisfied, since

\[ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = \frac{\partial}{\partial x} \left( \frac{\partial \psi}{\partial y} \right) + \frac{\partial}{\partial y} \left( -\frac{\partial \psi}{\partial x} \right) = \frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial^2 \psi}{\partial y \partial x} = 0 \]

Limitation of Stream Function

Two-Dimensional Flows of an Incompressible Fluid
Velocity Potential, $\phi$

$\phi = \phi(x, y, z, t)$

defined by

$u = \frac{\partial \phi}{\partial x}$; $v = \frac{\partial \phi}{\partial y}$; $w = \frac{\partial \phi}{\partial z}$

or

$\vec{u} = (u, v, w) = \text{grad } \phi = \nabla \phi$

In terms of $\phi$, the continuity equation is

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = \frac{\partial}{\partial x}\left(\frac{\partial \phi}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial \phi}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{\partial \phi}{\partial z}\right) =$$

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = \nabla^2 \phi = 0 \quad \text{(Laplace Eq.)}$$

$\nabla^2 = \nabla \cdot \nabla = \frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z} = \text{Laplace Operator}$

Limitation of Velocity Potential

For $\phi$ to make sense, we must have that

$$\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial \phi}{\partial y}\right) = \frac{\partial u}{\partial y} = \frac{\partial\phi}{\partial y} - \frac{\partial}{\partial y}\left(\frac{\partial \phi}{\partial x}\right) = \frac{\partial u}{\partial y}$$

Order of differentiation is immaterial!

$\frac{\partial u}{\partial y} \neq 0$  $\frac{\partial u}{\partial y} = 0$  at solid boundary $u = 0$

i.e. $\frac{\partial \phi}{\partial x} = 0$

So, for $\phi$ to exist, $\frac{\partial u}{\partial y} = 0$  unless fluid is inviscid ($\nu = 0$)
FLOW NET LIMITATIONS

Since flow nets are based on Stream Function and Velocity Potential concepts they are limited to:

Steady
Two-Dimensional Flows of an Incompressible and Inviscid Fluid,
or
Steady
Two-Dimensional Flows of an Ideal Fluid.

• We can still sketch flow nets for 3-D flows and get a very good physical picture of the nature of the flow, e.g. regions where velocities are large or small may be identified.

• The flow net allows fluid to have a slip-velocity along solid boundaries (they become streamlines!). Since there is no such thing as an ideal fluid the flow net features very close to solid boundaries are unreliable.

• Flow nets are useful when the bulk of the flow may be considered minimally affected by boundary shear stresses, and the flow is "converging", i.e. velocity is increasing along a streamline.
Stream Line Coordinates from Cartesian

\[ (u_s, u_s) = (\frac{\partial \psi}{\partial y}, -\frac{\partial \psi}{\partial x}) \]

\[ (u_s, u_s) = (\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}) \]

\[ \frac{\partial \psi}{\partial s} = \frac{\partial \psi}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial \psi}{\partial y} \frac{\partial y}{\partial s} = \frac{\partial \psi}{\partial x} (u_s \delta t) + \frac{\partial \psi}{\partial y} (u_s \delta t) = \]

\[ \int \left[ \frac{\partial \psi}{\partial x} \frac{\partial \psi}{\partial y} + \frac{\partial \psi}{\partial y} \left( -\frac{\partial \psi}{\partial x} \right) \right] \delta t = 0 \]

i.e.

\[ \frac{\partial \psi}{\partial s} = 0 \text{ along stream lines} \]

In the direction \( \perp s \), i.e. the "h" direction

\[ \frac{\partial \psi}{\partial h} = \frac{\partial \psi}{\partial x} (u_s^2 + u_s^2) \delta t = \frac{\partial \psi}{\partial x} (-u_s \delta t) + \frac{\partial \psi}{\partial y} (u_s \delta t) = \]

\[ \int \left[ \frac{\partial \psi}{\partial x} \left( -\frac{\partial \psi}{\partial x} \right) + \frac{\partial \psi}{\partial y} \left( \frac{\partial \psi}{\partial y} \right) \right] \delta t = (u_s^2 + u_s^2) \delta t \]

or

\[ \frac{\partial \psi}{\partial h} = (u_s^2 + u_s^2)^{1/2} = q_h \text{ - vel. in } s \text{-direct} \]

Similarly,

\[ \frac{\partial \phi}{\partial s} = \frac{\partial \phi}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial \phi}{\partial y} \frac{\partial y}{\partial s} = \frac{\partial \phi}{\partial x} (u_s^2 + u_s^2) \delta t \]

\[ \frac{\partial \phi}{\partial s} = (u_s^2 + u_s^2)^{1/2} = q_s \]

and

\[ \frac{\partial \phi}{\partial h} = \frac{\partial \phi}{\partial x} (-u_s \delta t) + \frac{\partial \phi}{\partial y} (u_s \delta t) = (-u_s u_s^2 + u_s u_s^2) \delta t = 0 \]

\[ \frac{\partial \phi}{\partial h} = 0 \]