We have derived volume conservation in terms of bulk flow description \( Q = VR = \text{constant} \), but if we need details we need to do a lot of work, e.g. draw flow nets to get details on velocity and then use Bernoulli to get details on pressure. Then we have to integrate \( p \) over a surface to get total pressure force. [and we don't get any information about shear forces!].

We want to get other quantities in terms of their bulk values, like \( Q \), but not the details. To do this we take: \text{Finite Control Volume}.

Let \( "M" \) be a fluid property per unit volume of fluid. With finite volume \( V \) we then have a total amount of "\( M \)"

\[
M = \int_V m \, dV
\]

The rate of change of \( M \) for this volume (consisting of the same molecules) is

\[
\frac{DM}{Dt} = \lim_{\Delta t \to 0} \frac{M(t + \Delta t) - M(t)}{\Delta t} = \text{Total or Material Derivative}
\]

where

\[
M_{\ast}(t) = \int_{V(t)} m(t) \, dV
\]

and

\[
M(t + \Delta t) = \int_{V(t + \Delta t)} m(t + \Delta t) \, dV = \int_{V(t)} \left[ m(t) + \frac{\partial m}{\partial t} \Delta t \right] \, dV
\]
\( V_0 = V(t_0) \quad M_0 = M(t_0) = \int_{V_0}^{} m_0 \, dV \)

\( V(t_0 + \Delta t) = V_0 - \Delta V_{in} + \Delta V_{out} \)

\( \Delta V_{in} = \int_{R_{in}} (q_1 \Delta t) \, dA = \left( \int_{R_{in}} q_1 \, dA \right) \Delta t \quad \Delta V_{out} = \int_{R_{out}} \left( q_1 \Delta t + \left( \frac{\partial m}{\partial t} \right) \Delta t + \Delta t^2 \right) \, dA \)

\( M(t_0 + \Delta t) = \int_{V_0}^{} m_0 \, dV + \left( \int_{R_{in}} \frac{\partial m}{\partial t} \, dA + \int_{R_{out}} \frac{\partial m}{\partial t} \, dA \right) \Delta t + \Delta t^2 \int_{R_{in}}^{R_{out}} \frac{\partial m}{\partial t} \, dA \)

\( M(t_0 + \Delta t) = \left( \text{"m" in } V_0 \text{ at } t_0 \right) + \Delta t \int_{R_{in}}^{R_{out}} \frac{\partial m}{\partial t} \, dA \)

\( \frac{DM}{Dt} = \int_{V_0}^{} \frac{\partial m}{\partial t} \, dV - \int_{R_{in}}^{} m q_1 \, dA + \int_{R_{out}}^{} m q_1 \, dA = \int_{V_0}^{} \frac{\partial m}{\partial t} \, dV - \int_{R_{in}}^{} m q_1 \, dA + \int_{R_{out}}^{} m q_1 \, dA \)
In Words
Rate of change of \( M \) for a volume following the fluid (some fluid particles within volume at all times) =

Rate of change of \( M \) within volume between fixed in- and outflow areas [the control volume] +
Rate of inflow of \( M \) into control volume -
Rate of outflow of \( M \) from control volume

Try this out for our old friend mass conservation.
Since \( \frac{dm}{dt} \) = mass/volume, and \( M \) as we move with the fluid is constant, we have

\[
\frac{dM}{dt} = 0 = \frac{2}{dV} \int_{V_1} q_d dV - \int_{V_1} q_{in} dA + \int_{V_1} q_{out} dA
\]

\[
M_{in} - M_{out} = \frac{dM}{dt}
\]

"old hat" (Lecture #5)

For volume itself - "\( V \)" = unity. If fluid is incompressible volume is conserved, and

\[
\frac{dV}{dt} = 0 = \frac{d}{dt} \left( \frac{dV}{dA} \right) - \int_{V_1} q_{in} dA + \int_{V_1} q_{out} dA
\]

\[
Q_{in} - Q_{out} = \frac{dV}{dt}
\]

"even older hat" (Lecture #5)
We can compact this expression, known as the **Reynolds Transport Theorem**

by the following trick:

\[ \vec{n} = \text{unit outward normal to } S_0 \]

\[ S_0 = A_s + A_p \]

At inflow
\[ \dot{Q}_l = -\vec{n} \vec{\dot{q}} \]

Along streamline, \( A_s \)
\[ \dot{Q}_l = \vec{n} \vec{\dot{q}} = 0 \]

At outflow
\[ \dot{Q}_l = \vec{n} \vec{\dot{q}} = \vec{Q}_l @ A_p \]

\[ \frac{DM}{Dt} = \frac{\partial}{\partial t} \iint \rho \vec{q} \, dt + \iiint \rho \vec{n} \cdot \vec{q} \, dS \]

Here's where we really need it: **Conservation of (Linear) Momentum**, or **Newton's Law**.

Rate of change of Momentum for a volume consisting of the same particles = Sum of Forces on this volume

Linear Momentum per unit volume = \( \rho \vec{q} \)

**Reynolds Transport Theorem**

Rate of change of momentum = \[ \frac{DM}{Dt} = \frac{\partial}{\partial t} \iint \rho \vec{q} \, dt + \iiint \rho \vec{q} (\vec{n} \cdot \vec{q}) \, dS = \Sigma (\text{Forces on } \vec{q}) \]