Homework Set #7

**Problem 1**
Consider a sequence of random variables \(X_1, X_2, \ldots, X_n, \ldots\), for example denoting the monthly profits of a supermarket chain. Suppose that \(X_i \sim (m, \sigma^2)\) for all \(i\) and that the correlation coefficient between \(X_i\) and \(X_j\), \(\rho_{ij}\), depends only on the time lag \(|i-j|\) as

\[
\rho_{ij} = 0.8^{|i-j|}
\]

Using conditional SM analysis, calculate and plot, as a function of \(k \geq 1\), the variances of \((X_{i+k}|X_i)\) and \((X_{i+k}|X_i,X_{i-1})\). Comment on the results.

**Problem 2**
\(X\) is an unknown quantity, say the compressive strength of a concrete column, with mean value \(m\) and variance \(\sigma^2\). Several indirect measurements of \(X\), in the form \(Z_i = X + \epsilon_i\) for \(i = 1, \ldots, n\), are made through a nondestructive technique. Under the assumption that the \(\epsilon_i\) are iid measurement errors with zero mean and common variance \(\sigma^2_{\epsilon}\), use conditional SM analysis to find the variance of \((X|Z_1,\ldots,Z_n)\). Plot this conditional variance against \(n\) for \(\sigma^2 = 1\) and \(\sigma^2_{\epsilon}\) either 1 or 0.2.

Useful result on the inverse of covariance matrices with a special “equicorrelated” structure. The inverse of an \(n \times n\) matrix \(\Lambda\) of the type:

\[
\Lambda = \sigma^2 \begin{bmatrix} 1 & \rho & \rho & \cdots & \rho \\ \rho & 1 & \rho & \cdots & \rho \\ \rho & \rho & 1 & \cdots & \rho \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \rho & \rho & \rho & \cdots & 1 \end{bmatrix}
\]

is:

\[
\Lambda^{-1} = \frac{1}{\sigma^2(1-\rho)[1+(n-1)\rho]} \begin{bmatrix} [1+(n-2)\rho] & -\rho & -\rho & \cdots & -\rho \\ -\rho & [1+(n-3)\rho] & -\rho & \cdots & -\rho \\ -\rho & -\rho & [1+(n-4)\rho] & \cdots & -\rho \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -\rho & -\rho & -\rho & \cdots & [1+(n-2)\rho] \end{bmatrix}
\]