Conditional Second-Moment Analysis

• Important result for jointly normally distributed variables $X_1$ and $X_2$

If $X_1$ and $X_2$ are jointly normally distributed with mean values $m_1$ and $m_2$, variances $\sigma_1^2$ and $\sigma_2^2$, and correlation coefficient $\rho$, then $(X_1 | X_2 = x_2)$ is also normally distributed with mean and variance:

$$
\begin{align*}
    m_{1|2}(x_2) &= m_1 + \rho \frac{\sigma_1}{\sigma_2} (x_2 - m_2) \\
    \sigma_{1|2}^2(x_2) &= \sigma_1^2 (1 - \rho^2)
\end{align*}
$$

(1)

Notice that the conditional variance does not depend on $x_2$.

The results in Eq. 1 hold strictly when $X_1$ and $X_2$ are jointly normal, but may be used in approximation for other distributions or when one knows only the first two moments of the vector $X = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$.

• Extension to many observations and many predictions

Let $X = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$, where $X_1$ and $X_2$ are sub-vectors of $X$. Suppose $X$ has multivariate normal distribution with mean value vector and covariance matrix:

$$
m = \begin{bmatrix} m_1 \\ m_2 \end{bmatrix}, \quad \Sigma = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix} \quad (\Sigma_{12} = \Sigma_{21}^T).
$$

Then, given $X_2 = x_2$, the conditional vector $(X_1 | X_2 = x_2)$ has jointly normal distributions with parameters:

$$
\begin{align*}
    m_{1|2}(x_2) &= m_1 + \Sigma_{12} \Sigma_{22}^{-1} (x_2 - m_2) \\
    \Sigma_{1|2}(x_2) &= \Sigma_{11} - \Sigma_{12} \Sigma_{22}^{-1} \Sigma_{21}
\end{align*}
$$

(2)

Notice again that $\Sigma_{1|2}$ does not depend on $x_2$.

As for the scalar case, Eq. 2 may be used in approximation when $X$ does not have multivariate normal distribution or when the distribution of $X$ is not known, except for the mean vector $m$ and covariance matrix $\Sigma$. 