1.204 Lecture 23

Analytic approximations
Vehicle routing
Transit design

**Analytic approximations**

- First spiral in developing problem solution
  - Assist in requirements, prototyping, initial results, review
  - Many analytic approximations are visual, unlike almost all algorithms
    - Recall role of visualization in finding roots of equations, and how poorly algorithms do without it
  - Generally allow a broader treatment of the question, with more variables, more flexible objectives and constraints
  - Provide guidance in framing heuristics
    - Many real problems do not have optimal algorithms
    - We have very few $O(n)$ or $O(n^2)$ algorithms for complex problems; most are $O(2^n)$
Vehicle routing

- Variables:
  - Number of routes or employees
  - Number of customers
  - Time windows or appointments
  - Capacity of vehicle or employee
  - Whether customers are known at start of route
  - And many others...

- Objectives
  - Customer service (timeliness, appointments)
  - Cost minimization

- Constraints
  - Labor rules, ...

Dispatch routing options

○ Work center

○ Customer
How to serve customers?

“Linehaul” path

Area a

Trip length $L$ to visit $n$ randomly distributed customers in area $a$ and return:

$$L = 2p + k \sqrt{na}$$

(Beardwood, Halton, Hammersley 1959)

Shape of dispatch zones

Which shape is better?
Zone shape

- Let’s try to elongate them
  - Tour length is the same for same number of points, same area, different shapes

Elongated zones have shorter driving distance if there are a lot of customers in each zone
Elongated zones, many customers

With many customers, elongated zones are better

Fat zones, few customers

If there are only a few customers, shape matters more. In this case, 'fat' zones are better.
Rules for building tours

- The “break point” for fat versus skinny zones is about 6 customers, based on simulation and geometric probability
  - If 6 or more customers can be served on a route:
    - Break up the area into skinny zones with the target number of points (6 or more)
    - Build tours in each zone, and fine tune
  - If 5 or fewer customers can be served on a route:
    - Break up the area into fat zones with the target number of points (5 or fewer). Only a few zones will touch the work center
    - Build tours in each zone, and fine tune
- Many dispatch systems ‘cluster’ jobs, which implicitly creates fat zones rather than skinny
  - Rack servicing has ~30-40 stops per day. Use skinny zones
  - Telecom dispatch has ~2-4 jobs per day. Use fat zones
  - Shared taxi has 2-4 stops per tour. Use fat zones
  - Dial-a-ride hopes to have 8-10 stops per tour. Use skinny zones

Rules for building tours-time windows

- Build skeleton elongated or fat routes (implicit zones) based on expected customer demand
  - Non-intersecting, non-overlapping routes
- Schedule stops in the following priority:
  - Tight time windows far from work center first
  - Then tight time windows near work center
  - Then other jobs far from work center
  - Then other jobs near work center
  - Pull next day’s work into today’s routes as feasible
  - Don’t give successive jobs with tight time windows to the same tech, if it can be avoided
- Rules determined from analysis and simulation
Summary - dispatch analysis

- Done before writing dispatch algorithm or system
  - Understand the problem, objectives and constraints
  - Use analytical optimization, simulation, probability, ...
  - Deal with broader set of issues than a single algorithm
  - Develop guidance for heuristics to be used

Transit system design

- Variables for bus system design:
  - Number of routes (route spacing)
  - Headway (frequency of service)
  - Fare
  - Vehicle size
  - Route length
  - Bus stop spacing
  - Express versus local service
  - Transfer pattern
Transit system design

- **Objectives:**
  - Maximize ridership
  - Minimize deficit (or maximize profit)
  - Equity in service levels

- **Constraints**
  - Available resources (deficit limit)
  - Minimum service levels
  - System capacity

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Route spacing and headway

Bus Line
Bus Stop
Boundary between users of different transit routes
Boundary of analysis area

Spatial variable in local area analysis

Figure by MIT OpenCourseWare.
Route spacing and headway

- General result over many objectives and constraints:
  - Optimal route spacing and headway are related by
    - $h^*= g^*/ 4jk$
  - At this point, average walk time= average wait time
  - Complications:
    - Ratio of wait time/headway, $k$, may vary with headway
    - There may be a ‘walk refusal distance’, and demand response to walk distance may be nonlinear
    - Headway varies over the day on a route: either choose an average spacing, or have peak-only routes
  - Option 2 on previous slide is ‘optimal’
    - Same operating cost and capacity as option 1
    - Better service (lower sum of walk and wait times)
    - Higher ridership
Fare and demand function

- Introduce a demand function:
  \[ \text{Mode share } t = (a_1 + a_2 (kh + (g+b)/4j)) + a_3 * d/v + a_4 * f + a_5 * d) \]
  - Where
    - \( a_1 \ldots a_5 \) are demand coefficients (\( a_5 \) is auto coefficient)
    - \( d \) is route distance, \( v \) is bus velocity, \( f \) is bus fare

- Total bus ridership in area \( P = T \)pXYt
  - Where
    - \( X,Y \) are dimensions (mi)
    - \( p \) is trip density (trips/mi^2/min)
    - \( T \) is time period (min)

Cost and objective function

- Bus operating costs:
  - \( C = 2 \) XYTc/ghv
  - There are \( X/g \) routes operating \( 2T/h \) trips of length \( Y/v \) at unit cost per minute of \( c \)

- Objective function: maximize net social benefits
  - Max consumers’ surplus \( G + \) revenue \( R (=Pf) – \) cost \( C \)
  - Subject to a deficit constraint \( (C-R \leq M) \)
  - \( G \) is a proxy for external benefits (air quality, GHG, …)
Overall formulation

- Max net benefit subject to \((\text{cost} - \text{revenue}) \leq M\)

\[
\begin{align*}
\max B_2 &= -TPXY(a_1 + a_3(kh + (g + b)/4j) \\
&\quad + a_2d/v + a_4f + a_5d)^2/2a_4 \\
\text{subject to} \\
2XTcY/ghv &= TPXY(a_1 + a_3(kh + (g + b)/4j) \\
&\quad + a_2d/v + a_4f + a_5d) - M \leq 0, \quad g, h, f \geq 0.
\end{align*}
\]

- Formulate using Lagrange multiplier

\[
\begin{align*}
\max B_1 &= -TPXY(a_1 + a_3(kh + (g + b)/4j) \\
&\quad + a_2d/v + a_4f + a_5d)^2/2a_4 - y_1(2XTcY/ghv) \\
&\quad - TPXY(a_1 + a_3(kh + (g + b)/4j) \\
&\quad + a_2d/v + a_4f + a_5d), \quad g, h, f \geq 0.
\end{align*}
\]

Solutions

- Take derivatives with \(f, g, h\) and \(y_2\) to obtain 4 nonlinear equations in 4 unknowns, and solve approximately:

\[
\begin{align*}
g^* &\approx (32j^2 k a_4 c(2y_2 - 1)/vp a_2 A y_2)^{1/3} \\
h^* &\approx (a_4 c(2y_2 - 1)/2jk^2 vp a_2 A y_2)^{1/3} \\
f^* &\approx [(1 - y_2)/(2y_2 - 1)] \\
&\quad \cdot [A/a_4 + (4ka_2^2(2y_2 - 1)/vpja_4^2 A y_2)^{1/3}].
\end{align*}
\]

- where
  - \(f\) is fare, \(g\) is route spacing, \(h\) is headway,
  - \(y_2\) is Lagrange multiplier or shadow price of $1 of benefit relative to $1 of deficit
- We vary \(y_2\) to get solutions ranging from min deficit \((y_2 = \text{infinite})\) to max social benefit \((y_2 = 1)\)
  - We can also add a vehicle capacity constraint \((y_3)\)
Model summary

- Model implemented in Java code
  - Download code and documentation
- Provides framework for designing bus system:
  - Routes, headways, fares, vehicle sizes, express/local service
  - Bus stop spacing (fewer are better)
  - Route circuity (less circuity is better)
  - (Model variation used in planning Logan Express)
- Allows variation in objective and constraints
- Provides insight before addressing detailed system design with actual network and routes, using optimization algorithms and simulation
  - Most of the the term was spent on optimization algorithms for decisions and design
  - Simulation not covered, used for truly difficult/detailed issues
- We’ll do analytical approximations for queuing systems in the next lecture
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