Lecture 3

Modeling Road Traffic Flow on a Link

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Lecture 3 Outline

- Time-Space Diagrams and Traffic Flow Variables
- Introduction to Link Performance Models
- Macroscopic Models and Fundamental Diagram
- Volume-Delay Function
- (Microscopic Models: Car-following Models)
- Relationship between Macroscopic Models and Car-following Models
- Summary
Time-Space Diagram: Analysis at a Fixed Position

Flows and Headways

- $m(x)$: number of vehicles that passed in front of an observer at position $x$ during time interval $[0,T]$. (ex. $m(x)=5$)

- Flow rate: $q(x) = \frac{m(x)}{T}$

- Headway $h_j(x)$: time separation between arrival time of vehicles $i$ and $i+1$

- Average headway: $\bar{h}(x) = \frac{\sum_{j=1}^{m(x)} h_j(x)}{m(x)}$

- What is the relationship between $q(x)$ and $\bar{h}(x)$?
Flow Rate vs. Average Headway

- If $T$ is large, $T \approx \sum_{j=1}^{m(x)} h_j(x)$

- Then, $q(x) = \frac{T}{m(x)} \approx \frac{\sum_{j=1}^{m(x)} h_j(x)}{m(x)} = \bar{h}(x)$

  $\Rightarrow q(x) \approx \frac{1}{\bar{h}(x)}$  This is intuitively correct.

- $q(t)$ is also called **volume** in traffic flow system circles (i.e. 1.225)

- $q(t)$ is also called **frequency** in scheduled systems circles (i.e. 1.224)

Time-Space Diagram: Analysis at Fixed Time

A diagram illustrating a time-space analysis at fixed time, showing position vs. time with a fixed location at $s_1$ and another at $s_2$.
Density and Average Spacing

- \( n(t) \): number of vehicles in a road stretch of length \( L \) at time \( t \)
- Density: \( k(t) = \frac{n(t)}{L} \)
- \( s_i(t) \): spacing between vehicle \( i \) and vehicle \( i+1 \)

\[
L \approx \sum_{i=1}^{n(t)} s_i(t)
\]

\[
\frac{1}{k(t)} = \frac{L}{n(t)} \approx \frac{\sum_{i=1}^{n(t)} s_i(t)}{n(t)} = \bar{s}(t)
\]

\[
k(t) = \frac{1}{\bar{s}(t)} \quad \text{(Is this intuitive?)}
\]

Performance Models of Traffic on a Road Link

- Link: a representation of a highway stretch, road from one intersection to the next, etc.
- Example of measures of performance:
  - Travel time
  - Monetary or environmental cost
  - Safety
- Main measure of performance: travel time
- 3 types of models:
  - Macroscopic models: Fundamental diagram (valid in static (stationary) conditions only. Long roads and long time periods)
  - Microscopic models: Car-following models (no lane changes)
  - Volume-delay functions
Macroscopic Flow Variables

- Three macroscopic flow variables of a link:
  - Average density \( k \) (also denoted by \( \rho \))
  - Average flow \( q \)
  - Average speed \( u \) (also denoted \( v \))

- Relationships between variables:
  - \( q = uk \)
  - \((k,q)\) curve: **Fundamental diagram**
    - Fundamental diagram is a property of the road, the drivers and the environment (icy, sunny, raining)

- 3 variables + 2 equations \( \Rightarrow \) only one variable can be an independent variable (But one of the variables \((k,u,q)\) can not be independent)

Data Collected from Holland Tunnel (Eddie, 63)

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<th>Speed (km/hr)</th>
<th>Average Spacing (m)</th>
<th>Concentration (veh/km)</th>
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1.225, 11/01/02 Lecture 3, Page 9
(Density, Speed) Diagram for the Field Data

(Density, Speed) Diagram with a Fitted Curve

\[ y = 0.0102x^2 - 1.7549x + 84.144 \]
(Density, Flow) Diagram from the Field Data

\[ y = 0.0076x^3 - 1.4481x^2 + 74.248x + 80.889 \]

(Density, Flow) Diagram with a Fitted Curve
(Flow, Speed) Diagram from the Field Data

![Flow vs Speed Diagram](image)

(Spacing, Speed) Diagram from the Field Data

![Spacing vs Speed Diagram](image)
(Flow, Pace) Diagram from the Field Data

\begin{figure}
\centering
\includegraphics[width=\textwidth]{flow_pace_diagram.png}
\caption{(Flow, Pace) Diagram from the Field Data}
\end{figure}

1.225, 11/01/02
Lecture 3, Page 17

Relationships between Flow Variables

\begin{figure}
\centering
\includegraphics[width=\textwidth]{flow_variables.png}
\caption{Relationships between Flow Variables}
\end{figure}

1.225, 11/01/02
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- $k_{jam}$: jam density (the highway stretch is like a parking lot!)
- $k_{jam}^{-1}$ = a car length
- $q = uk$
- $q_{max} = q(k_c)$ is the maximum flow, or link capacity
- $u_c = u(k_c) = \frac{q_{max}}{k_c}$
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Fundamental Diagram

- \( k \in [k_c, k_{jam}] \): arise when flow is slower down stream due to lane drops, a slow plowing-truck, etc
- \( k_c \) is critical, since it marks the start of an “unstable” flow area where additional input of cars decrease flow served by the highway
- \((k, q)\) diagram is fundamental since it represents the three variable as compared to the other diagrams

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Derived Diagrams

- In general, \( q \) cannot be used as a variable (why?)
- In the road network planning area:
  - \( q \) is also called volume
  - travel time is also called travel “delay”
  - In the case of volume-delay functions, \( q \) is used as a variable
Examples of Classical Volume-Delay Functions

- **Notation:**
  - $q$ is the link flow
  - $t(q)$ is the link travel time
  - $c$ is the **practical capacity**
  - $\alpha$ and $\beta$ are calibration parameters

- **Davidson’s function:**
  \[ t(q) = t(0)[1 + \alpha \frac{q}{c - q}] \]

- **US Bureau of Public Roads**
  \[ t(q) = t(0)[1 + \alpha \frac{q}{c}]^\beta \]

Observations on Classical Volume-Delay Functions

- **Examples where the classical model may be acceptable:**
  - Delay at a signalized link
  - $q < q_{\text{max}}$ (mild congestion)

- **What makes the classical model interesting?**
  - It is a function (There is only one value for a given $q$)
  - Typical functions used are increasing with $q$, and their derivatives are also increasing (“it holds water” $\iff$ it is convex)
  - The above are analytical properties that have been adopted to study the properties of, and design solution algorithms for, network traffic assignment models (Lectures 4-6)
  - $\implies$ An example of tradeoffs made between realism and computational tractability
Link Travel Time Models: Car-Following Models

- Notation:

\[ x_n(t) - x_{n+1}(t) = \text{spacing (space headway)} = l_{n+1}(t) + \frac{1}{k_{\text{jam}}} \]

- Speed of vehicle \( n \):
  \[ \frac{dx_n(t)}{dt} = \dot{x}_n(t) \]

- Acceleration (deceleration) of vehicle \( n \):
  \[ \frac{d\dot{x}_n(t)}{dt} = \ddot{x}_n(t) \]

- Car-following regime: \( l_{n+1}(t) \) is below a certain threshold

Simple car-following model:

\[ \ddot{x}_{n+1}(t + T) = a \dot{x}_{n+1}(t) = a(\dot{x}_n(t) - \dot{x}_{n+1}(t)) \]

\( T \): reaction time \( (T \approx 1.5 \text{ sec}) \)

\( a \): sensitivity factor \( (a \approx 0.37 \text{ s}^{-1}) \)

Questions about this simple car-following model:

- Is it realistic?
- Does it have a relationship with macroscopic models?
From Microscopic Models To Macroscopic Models

- Simple car-following model: \( \ddot{x}_{n+1}(t) = a(\dot{x}_n(t) - \dot{x}_{n+1}(t)) \) \( (T = 0) \)
- Fundamental diagram: \( q = q_{\text{max}} \left( 1 - \frac{k}{k_{\text{jam}}} \right) \)
- Proof of “equivalency”
  \[
  \ddot{x}_{n+1}(y) = a(\dot{x}_n(y) - \dot{x}_{n+1}(y))
  \]
  \[
  \ddot{x}_{n+1}(y)dy = a(\dot{x}_n(y) - \dot{x}_{n+1}(y))dy = a\dot{x}_{n+1}(t)dy
  \]
  \[
  \int_0^1 \ddot{x}_{n+1}(y)dy = \int_0^1 a\dot{x}_{n+1}(t)dy
  \]
  \[
  u_{n+1}(t) - u_{n+1}(0) = a(l_{n+1}(t) - l_{n+1}(0))
  \]
  \[
  u_{n+1}(t) = a l_{n+1}(t) + u_{n+1}(0) - al_{n+1}(0)
  \]
  If \( l_{n+1}(t) = 0 \), then \( u_{n+1}(t) = 0 \Rightarrow u_{n+1}(0) - al_{n+1}(0) = 0 \)

From Microscopic Model to Macroscopic Model

\[
 u_{n+1}(t) = al_{n+1}(t) = a \left( \frac{1}{k_{n+1}(t)} - \frac{1}{k_{\text{jam}}} \right)
\]
\[
 \Rightarrow u = a \left( \frac{1}{k} - \frac{1}{k_{\text{jam}}} \right)
\]
\[
 \Rightarrow q = uk = a \left( \frac{1}{k} - \frac{1}{k_{\text{jam}}} \right) k = a \left( 1 - \frac{k}{k_{\text{jam}}} \right)
\]

If \( k = 0 \), then \( q = a \)
Since \( q = a \geq a \left( 1 - \frac{k}{k_{\text{jam}}} \right) \), then \( a = q_{\text{max}} \)
\[
 \Rightarrow q = q_{\text{max}} \left( 1 - \frac{k}{k_{\text{jam}}} \right)
\]

- Note: if \( k \to 0 \), then \( u \to \infty \). Does this make sense?
Non-linear Car-following Models

\[
x_{n+1}(t + T) = a_0 \frac{\dot{x}_n(t) - \dot{x}_{n+1}(t)}{(x_n(t) - x_{n+1}(t))^5}
\]

\[
= a_0 \frac{\dot{x}_n(t)}{\left[ l_{n+1}(t) + \frac{1}{k_{jam}} \right]^{1.5}}
\]

If \( T = 0 \), the corresponding fundamental diagram is:

\[
q = u_{\text{max}} k \left[ 1 - \left( \frac{k}{k_{\text{jam}}} \right)^{0.5} \right]
\]

Flow Models Derived from Car-Following Models

\[
\ddot{x}_{n+1}(t + T) = a_0 \dot{x}_{n+1}^m(t + T) \frac{\dot{x}_n(t) - \dot{x}_{n+1}(t)}{(x_n(t) - x_{n+1}(t))^5}
\]

<table>
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<tr>
<th>( l )</th>
<th>( m )</th>
<th>Flow vs. Density</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>( q = u_{\text{max}} ) ( \left[ 1 - \frac{k}{k_{\text{jam}}} \right] )</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>( q = u_{\text{max}} k \ln \left( \frac{k_{\text{jam}}}{k} \right) )</td>
</tr>
<tr>
<td>1.5</td>
<td>0</td>
<td>( q = u_{\text{max}} \left[ 1 - \left( \frac{k}{k_{\text{jam}}} \right)^{1.5} \right] )</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>( q = u_{\text{max}} \left[ 1 - \frac{k}{k_{\text{jam}}} \right] )</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>( q = u_{\text{max}} k \exp \left[ 1 - \frac{k}{k_{\text{jam}}} \right] )</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>( q = u_{\text{max}} k \exp \left[ \frac{1}{2} \left( \frac{k}{k_{\text{jam}}} \right)^2 \right] )</td>
</tr>
</tbody>
</table>
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