Lecture 5

Assignment on Traffic Networks

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Lecture 5 Outline

- Summary from previous lectures:
  - Assignment on non-congested networks: All-or-nothing assignment
  - Volume-delay functions for “congested” networks
- Framework for static traffic assignment models
- Static traffic assignment: concepts
- Static traffic assignment: principles
- User Optimal (UO) and System Optimal (SO) static traffic assignment
- Summary
Non-Congested Road Network and O-D Matrix

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In R7, \( t(i, j) \) means demand for O-D pair \((i, j)\). \( t(i, j) \) does not mean travel time of link \((i, j)\).

Other common notation: \( q(i, j) \) or \( q_{ij} \)

Assign O-D Flows Originating from Node 1

Shortest path tree from node 1

Assign flows originating from node 1
Assign O-D Flows Originating from Node 2

Shortest path tree from node 2

Assign flows originating from node 2

Assign O-D Flows Originating from Node 3

Shortest path tree from node 3

Assign flows originating from node 3
Assign O-D Flows Originating from Node 4

Assign flows originating from node 4

Shortest path tree from node 4

Assign O-D Flows Originating from Node 5

Assign flows originating from node 5

Shortest path tree from node 5
Results of All-or-Nothing Assignment

- All-or-nothing (AON) assignment does not consider congestion
- The solution may not be unique (Why? Is unique solution important?)
- AON assignment does not make sense if certain links are congested
- To account for congestion, travel time must depend on link flow
- How to change the AON assignment method to make it work?
  ⇒ Equilibrium traffic assignment

Derived Diagrams from the Fundamental Diagram

- In general, $q$ cannot be used as a variable (why?)
- In the traffic planning area:
  - $q$ is also called volume
  - travel time is also called travel delay
  - In the case of volume-delay functions, $q$ is used as a variable
Framework for Static Traffic Assignment Models

- Conceptual framework:
  - Network representation of the structure of a physical road network
  - Link performance functions

- Principles of assignment to represent the supply/demand interaction
  - User Optimal (U.O.): O-D flows are assigned to paths with minimum travel time
  - System Optimal (S.O.): O-D flows are assigned such that total travel time on the network is minimum

Example

Highway (Link 1)
\[ t_1(x) = 10 + x \]

Small On-Ramp (Link 3)
\[ t_3(x) = 0 \]

Arterial Street (Link 2)
\[ t_2(x) = 90 + x \]

\[ q_{ac} \Rightarrow a \quad c \Rightarrow q_{ac} \]
\[ q_{bc} \Rightarrow b \quad b \Rightarrow q_{bc} \]
Traffic Assignment Concepts

- Conceptual Network
  - Demand
    - O - Ds: (a, c) and (b, c)
    - (a, c): \(q_{ac}\) vehicles/hr
    - (b, c): \(q_{bc}\) vehicles/hr

- Path flow variables
  - O - D (a, c): one path
    - Link 1: \(f_{1ac}\)
  - O - D (b, c): two paths
    - Link 2: \(f_{1bc}\)
    - Link 3: \(f_{2bc}\)

- O-D flows and path flows
  - \(q_{ac} = f_{1ac}\)
  - \(q_{bc} = f_{1bc} + f_{2bc}\)

Traffic Assignment Concepts (cont.)

- Link (arc) flows and path flows
  - \(x_1 = f_{1ac} + f_{2bc}\)
  - \(x_2 = f_{1bc}\)
  - \(x_3 = f_{2bc}\)

- Arc-path incidence matrix
  - O-D → a \(\rightarrow\) c \(\rightarrow\) b \(\rightarrow\) c
  - Path → 1 \(\rightarrow\) 1 \(\rightarrow\) 2
  - Link → 1 \(\rightarrow\) 0 \(\rightarrow\) 1
  - 2 \(\rightarrow\) 0 \(\rightarrow\) 1
  - 3 \(\rightarrow\) 0 \(\rightarrow\) 1

- Assume that \(f_{2bc} = pq_{bc}\)

- Assignment matrix
  - \[
  \begin{bmatrix}
  x_1 \\ x_2 \\ x_3
  \end{bmatrix} =
  \begin{bmatrix}
  1 & p \\ 0 & 1 - p \\ 0 & p
  \end{bmatrix}
  \begin{bmatrix}
  q_{ac} \\ q_{bc}
  \end{bmatrix}
  \]
Traffic Assignment Concepts (cont.)

- $t_1, t_2, t_3$ are the travel times of links 1, 2, 3.
- "Congested" networks:
  - Link travel times depend on link flows
  - Example:
    \[ t_1(x_1) = 10 + x_1, \quad t_2(x_2) = 90 + x_2, \quad t_3(x_3) = 0 \]
- Path travel-times as a function of link travel-times:
  \[ C_{1}^{bc} = t_1, \quad C_{2}^{bc} = t_2, \quad C_{3}^{bc} = t_3 \]
- Total travel times (What is the unit of this quantity?):
  \[ x_1 \times (10 + x_1) + x_2 \times (90 + x_2) + x_3 \times (0) \]

Assignment Principles

- If OD flows are infinitely divisible:
  - There is an infinite number of assignments
  - Which assignment should we choose?
    \[ \Rightarrow \text{We need additional assumptions to define a less ambiguous assignment} \]
- Assignment principle: a principle used to determine an assignment
- Examples of assignment principles:
  - **User-optimal**: between each O-D pair, all used paths have equal and minimum travel times
  - **System-optimal**: the total travel times are minimum
Mathematical Expressions of Assignment Principles

- User-optimal traffic assignment principle: find $p$ such that:
  - O-D (b, c):
    - If $p = 0$, $(t_1 + t_3) >= t_2$
    - If $p = 1$, $(t_1 + t_3) <= t_2$
    - If $0 < p < 1$, $t_1 + t_3 = t_2$
  - O-D (a, c): the question is not posed as there is only one path
- System optimal: find $p$ that minimizes:
  
  $$f_1^{ac} C_1^{ac} + f_1^{hc} C_1^{hc} + f_2^{ac} C_2^{ac} = x_1 t_1 + x_2 t_2 + x_3 t_3$$

Solution of the U.O. Assignment

- We want to solve for $p \in [0,1]$ such that:
  - If $p = 0$, $(t_1(x_1) + t_3(x_3)) >= t_2(x_2)$
  - If $p = 1$, $(t_1(x_1) + t_3(x_3)) <= t_2(x_2)$
  - If $0 < p < 1$, $t_1(x_1) + t_3(x_3) = t_2(x_2)$
- and

  $$q_{ac} (= 80) \quad t_1(x_1) = 10 + x_1 \quad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 & p \\ 0 & 1-p \end{bmatrix} \begin{bmatrix} q_{ac} \\ q_{hc} \end{bmatrix}$$

  $$q_{hc} (= 10) \quad t_2(x_2) = 90 + x_2 \quad t_3(x_3) = 0$$
Solution of the U.O. Assignment (cont.)

- \( t_2(x_2) - (t_1(x_1) + t_3(x_3)) = (90 + x_2) - (10 + x_1) = 80 + x_2 - x_1 \)
- \( x_2 - x_1 = (q_{bc} - x_3) - (q_{ac} + x_3) = (q_{bc} - q_{ac}) - 2x_3 \)
- \( t_2(x_2) - (t_1(x_1) + t_3(x_3)) = 80 + q_{bc} - q_{ac} - 2x_3 \)
  \[ \text{If } \frac{80 + q_{bc} - q_{ac}}{2} \geq 0, \text{ then } x_3 = \frac{80 + q_{bc} - q_{ac}}{2} \]

Example: \((q_{ac}, q_{bc}) = (80,10) \rightarrow (x_1, x_2, x_3) = (85,5,5)\)

- The travel time on any path is: 95
- Total travel time (per hour): \((80+10)*95 = 8550 \text{ veh-hrs (per hour)}\)

(UO) Assignment May Depend on the Demand

- \( \text{If } \frac{80 + q_{bc} - q_{ac}}{2} \geq 0, \text{ then } (x_1, x_2, x_3) = \left( \frac{80 + q_{bc} + q_{ac}}{2}, \frac{q_{bc} - 80 + q_{ac}}{2}, \frac{80 + q_{bc} - q_{ac}}{2} \right) \)
- \( \text{If } \frac{80 + q_{bc} - q_{ac}}{2} < 0, \text{ then } (x_1^*, x_2^*, x_3^*) = (q_{ac}, q_{bc}, 0) \)

Proof: \( p^* = 0 \) and \( t_2(q_{bc}) - (t_1(q_{ac}) - t_1(0)) = \frac{80 + q_{bc} - q_{ac}}{2} < 0 \)

- The U.O. assignment is a function of the demand, and the function may be non-linear. Example:
  \((q_{ac}, q_{bc}) = (80,10) \rightarrow (x_1^*, x_2^*, x_3^*) = (85,5,5)\)
  \((q_{ac}, q_{bc}) = (160,20) \rightarrow (x_1^*, x_2^*, x_3^*) = (160,20,0)\)
  \((160,20) = 2 \times (80,10) \text{ But } (160,20,0) \neq 2 \times (85,5,5)\)

Remark: An increase in demand may lead to a decrease in the flow on a link
Building More Roads Is Not Always Better

- Without Link 3, there is only one possible assignment
  \((x_1^*, x_2^*) = (q_{ac}, q_{bc}) = (80, 10)\)
  Total travel times in one hour: \(80 \times (10+80) + 10 \times (90+10) = 8200\) veh-hr (in one hour)
- U.O. with Link 3:
  Total travel times (in one hour): \(8550\) veh-hr (in one hour)
- System travel times are worse if one adds Link 3! (Is this intuitive?)
- This phenomenon is known as Braess “Paradox”, and is not an isolated phenomenon.

System Optimal Assignment

- \[\min \quad x_1 t_1(x_1) + x_2 t_2(x_2) + x_3 t_3(x_3)\]
- \[s.t. \quad f_1^{aw} = q_{ac}\]
  \[f_1^{bc} + f_2^{bc} = q_{bc}\]
  \[f_1^{aw} \geq 0, f_1^{bc} \geq 0, f_2^{bc} \geq 0\]
  Non-negativity
- \[x_1 = f_1^{aw} + f_1^{bc}\]
  \[x_2 = f_1^{bc}\]
  \[x_3 = f_2^{bc}\]
- S.O. solution: \((x_1^*, x_2^*, x_3^*) = (80, 10, 0)\), if \((q_{ac}, q_{bc}) = (80, 10)\)
- \[\min x_1 t_1(x_1) + x_2 t_2(x_2) + x_3 t_3(x_3) = \min \int_0^d m_1(x_1) dx_1 + \int_0^d m_2(x_2) dx_2 + \int_0^d m_3(x_1) dx_3\]
  Where: \[m_1(x_1) = \frac{d(x_1 t_1(x_1))}{dx_1}, \quad m_2(x_2) = \frac{d(x_2 t_2(x_2))}{dx_2}, \quad m_3(x_1) = \frac{d(x_3 t_3(x_1))}{dx_1}\]
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- Volume-delay functions for “congested” networks and static demand
- Framework for static traffic assignment models
- Static traffic assignment concepts and principles
- User Optimal and System Optimal static traffic assignment
- Summary