Spreading of lava on a horizontal plane. Huppert, (1986).

Let a finite mass of lava is initially released on a horizontal plane and spread slowly in all radial directions. Invoke the lubrication approximation and assume the local radial velocity to be

\[ u(r, z, t) = -\frac{g}{2\nu} \frac{\partial h}{\partial r} z(2h - z) \]  

Then use the law of mass conservation

\[ \frac{\partial h}{\partial t} + \frac{1}{r} \frac{\partial}{\partial r} \left( r \int_{0}^{h} u \, dz \right) = 0 \]  

(2)

to show that

\[ \frac{\partial h}{\partial t} = \frac{g}{3\nu} \frac{1}{r} \left( r \frac{\partial h}{\partial r} \right) \]  

(3)

Show also that

\[ 2\pi \int_{0}^{R(t)} rh(r, t) \, dr = \text{constant} = V \]  

(4)

where \( R(t) \) is the front of the spreading lava. The boundary conditions are

\[ h(R(t), t) = 0, \quad \text{and} \quad \frac{\partial h(0, t)}{\partial r} = 0 \]  

(5)

Show that the similarity solution exists and is of the form

\[ h(r, t) = A \frac{t^{1/4}}{r^{1/8}} f(\eta), \quad \text{with} \quad \eta = \frac{Cr}{t^{1/8}} \]  

(6)

subject to the integral constraint (4).

Derive the governing equation and boundary conditions for \( f(\eta) \) and adjust the constants \( A \) and \( C \) so that the governing equations look the simplest. What condition determines \( \eta_R \)?

Try to solve the problem analytically.