1. **Hot spring at Yellowstone** In a heated rock there is a fracture in the shape of a tube of circular cross section of radius $a$. The axis of the tube is a semicircular arc of large radius $R \gg a$. Referring to figure 1 both ends of the fracture pierce through the ground surface. At one end $A$ water enters the fracture from a reservoir at depth $h$ and shoots out at the other end $B$.

Let the geothermal temperature in the rock be varying linearly with depth:

$$ T_R = T_o (1 - b z) = T_o (1 + b R \sin \theta) \quad (1) $$

where $b$ is a positive constant, Derive an approximate theory for the cross-sectional averaged temperature of water at the exit. Identify the the central dimensional parameter in the problem, and plot the exit temperature as a function of this parameter. This problem may suggest ways to extract geothermal energy from hydraulic fractures.

**Remarks:**

To go through the formal procedure of normalization, etc., would take too much time. Try to get to an approximation quickly by, ignoring the effects of curvature whenever possible.
(since $a/R \ll 1$), assuming that the flow is driven by pressure gradient alone, but the fluid temperature in the crack depends on the flow.

**Suggested steps**

- Ignore the curvature of the tube, find the velocity distribution $u_\theta(r)$ in the tube due to the pressure head difference $\rho gh$.

- Let the fluid temperature be

  $$T_f(r, \theta) = T_R(\theta) + T'(r, \theta)$$

Invoking linearity if necessary and solve first for the radial dependence of $T'$, then take the cross-sectional average, e.g.,

$$\bar{T}_f = \frac{1}{\pi a^2} \int_0^a 2\pi r T_f dr \quad (3)$$

- Use integrated energy balance to derive a differential equation for $\bar{T}_f$.

- Recall that the ODE

  $$\frac{dy}{dx} + hy = g(x) \quad (4)$$

  can always be solved by rewriting it as

  $$e^{-hx} \frac{d}{dx} (y e^{hx}) = g(x) \quad (5)$$