4.2 Approximations for small temperature variation

4.2.1 Mass conservation and almost incompressibility

Recall the law of mass conservation:

$$-\frac{1}{\rho} \frac{D\rho}{Dt} = \nabla \cdot \vec{q}$$

Let the time scale be $L/U$. The left-hand-side is of the order $\frac{U\Delta \rho}{L\rho}$ while the right-hand-side is $\frac{U}{T}$. For $\Delta T = O(10^\circ C)$, their ratio is

$$\frac{\Delta \rho}{\rho} \sim \frac{\Delta T}{T} \sim \frac{10^\circ K}{300^\circ K} \ll 1$$

Therefore,

$$\nabla \cdot \vec{q} = 0. \quad (4.2.1)$$

The fluid is approximately incompressible even if $\Delta T \neq 0$.

4.2.2 Momentum conservation and Boussinesq approximation

In static equilibrium $\vec{q}_o \equiv 0$. Therefore,

$$-\nabla p_o + \vec{f} \rho_o = 0. \quad (4.2.2)$$

Let $p = p_d + p_o$ where $p_d$ is the dynamic pressure

$$\rho = \rho_d + \rho_o$$

$$-\nabla p + \rho \vec{f} = -\nabla p_o + \rho_o \vec{f} - \nabla p_d + (\rho - \rho_o) \vec{f}$$

Therefore,

$$\rho \frac{D\vec{q}}{Dt} = -\nabla p_d + \nabla \cdot \vec{\tau} + (\rho - \rho_o) \vec{f} \quad \text{buoyancy force} \quad (4.2.3)$$

Now

$$\rho = \bar{\rho}_o[1 - \beta(\Delta T_o + \Delta T_d)] \quad (4.2.4)$$
Hence

\[ \rho_o = \bar{\rho}_o(1 - \beta \Delta T_o), \quad \rho_d = -\bar{\rho}_o \beta \Delta T_d, \]

and

\[ (\rho - \rho_o) \bar{f} = -\bar{\rho}_o ( -g ) \beta \Delta T_d \bar{k} = \bar{\rho}_o \beta \Delta T_d \bar{k} \]

(4.2.5)

For mildly varying \( \rho_o \) and small \( \rho - \rho_o \), we ignore the variation of density and approximate \( \rho_o \) by a constant everywhere, except in the body force. This is called the Boussinesq approximation. Thus

\[ \bar{\rho}_o \frac{D \bar{q}}{D t} = -\nabla p_d + \nabla \cdot \bar{\tau} + \bar{\rho}_o \beta \Delta T_d \bar{k} \]

(4.2.6)

where

\[ \bar{\rho}_o = \rho_o (z = 0) \]

4.2.3 Total energy

Using Eqn. (4.2.1) in Eqn. (??) and the Boussinesq approximation

\[ \bar{\rho}_o C \frac{D T}{D t} = \frac{\partial}{\partial x_i} K \frac{\partial T}{\partial x_i} + \Phi \]

(4.2.7)

Here \( T \) is the total temperature (static + dynamic).

Now

\[ \frac{\Phi}{\bar{\rho}_o C \frac{D T}{D t}} \sim \frac{\mu U^2 / L^2}{\bar{\rho}_o C \frac{\partial T}{\partial x_i}} \sim \frac{\mu}{\bar{\rho}_o U L} \frac{U^2}{C \Delta T} = \frac{E}{Re} \]

where

\[ E = \frac{U^2}{C \Delta T} = \text{Eckart No.}, \quad Re = \frac{\rho U L}{\mu} = \text{Reynolds No.} \]

In environmental problems, \( \Delta T \sim 10^6 K, L \sim 10 m, U \sim 1 \text{m/sec} \), the last two columns of Table 4.1: Typical values for air and water

<table>
<thead>
<tr>
<th></th>
<th>Water</th>
<th>Air</th>
</tr>
</thead>
<tbody>
<tr>
<td>C (erg/s-gr^{-2}K)</td>
<td>4 × 10^7</td>
<td>10^7</td>
</tr>
<tr>
<td>K (ergs-cm^{-2}K)</td>
<td>0.6 × 10^5</td>
<td>0.3 × 10^5</td>
</tr>
<tr>
<td>( \nu )(cm^2/s)</td>
<td>10^2</td>
<td>2 × 10^{-2}</td>
</tr>
<tr>
<td>( \beta )(1/°K)</td>
<td>10^{-3}</td>
<td>1/300</td>
</tr>
<tr>
<td>( E )</td>
<td>0.25 × 10^{-2}</td>
<td>10^{-4}</td>
</tr>
<tr>
<td>( Re )</td>
<td>10^5</td>
<td>0.5 × 10^5</td>
</tr>
</tbody>
</table>

Table 4.1 is obtained. Hence \( \Phi \) is negligible, and

\[ \bar{\rho}_o C \frac{D T}{D t} = \frac{\partial}{\partial x_i} K \frac{\partial T}{\partial x_i} \]

(4.2.8)
Only convection and diffusion are dominant. This is typical in natural convection problems.

**Remark 1.** In many engineering problems (aerodynamics, rocket reentry, etc.), heat is caused by frictional dissipation in the flow, therefore, $\Phi$ is important. These are called *forced convection* problems. In environmental problems, flow is often the result of heat addition. Here the flow problems are referred to as the **natural convection**.

**Remark 2:** Since $\bar{T}$ appears as a derivative only, only the variation of $T$, i.e., the difference $T - \bar{T}_o$ matters, where $\bar{T}_o$ is a reference temperature.

**Remark 3:** In turbulent natural convection

$$u = \bar{u} + u' \quad T = \bar{T} + T'$$

(A.2.9)

Averaging Eqn. (4.2.8)

$$\bar{\rho}c \frac{D\bar{T}}{Dt} = -\bar{\rho}c \frac{\partial \bar{u}i\bar{T}'}{\partial x_i} + \frac{\partial}{\partial x_i} K \frac{\partial T'}{\partial x_i}$$

(A.2.10)

heat flux by turbulence

If the correlation term is modeled as eddy diffusion, the form would be similar to (4.2.8).